

'A NUCLEAR STATISTICAL MODEL WITH APPLICATIONS  
TO STABILITY'.

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## 1. INTRODUCTION.

Leaving out of consideration those nuclei of small atomic number, it is possible to develop a statistical theory of nuclei applicable to the nuclei of medium and large atomic number. BETHE and BACHER (B.1.) have treated this subject in great detail, and a resumé of their work is given in chapter 2. Their treatment is not as general as possible because they have not taken into account the four different types of forces between nuclear particles, namely, the Majorana, Heisenberg, Bartlett, and Wigner forces. The binding energies which BETHE and BACHER obtain are not in good agreement with experimental results. Also there is no justification for the use of either the interaction potentials proportional to  $\exp(-r^2/a^2)$  or to  $\exp(-r/a)$ .

From the theory of the heavy electron, it is deduced that the correct form of the nuclear interaction potential is proportional to  $(a/r)\exp(-r/a)$ .  $a$  is a constant giving the width of the potential hole, i.e.  $a$  gives the effective range of the nuclear interaction forces. If  $B$  is the constant of proportionality giving the depth of the potential hole, then  $B$  determines the strength of the interaction forces.

I have attempted to develop the statistical theory with an interaction potential of this form, but it also gives results in disagreement with the experimental evidence on binding energies. If I try to fit the theoretical equations to any particular nucleus, so that the binding energy is the same as the experimental value, I obtain a value of  $a$  much greater than the value of  $a$  as deduced from the heavy electron theory. I have, therefore, fixed the value of  $a$  from the heavy electron theory and taken the interaction potential to be

$$B(a/r + k)\exp(-r/a),$$

i.e. a linear combination of the heavy electron potential and the potential proportional to  $\exp(-r/a)$ . For small values of  $r$  the most important term is the heavy electron potential.

I take into consideration the four different types of forces already enumerated. For reasons advanced in chapter 5 I assume that the Bartlett force is twice the Wigner force. This assumption also simplifies the mathematical treatment considerably. In addition the assumption receives some confirmation from the similar result obtained by VOLZ (V.1.).

The spin of the nuclear particles is introduced by using a total wave function for the nucleus suggested by FUCHS (unpublished). The eigenfunctions of the individual particles of the nucleus are then taken to be plane waves (Chapter 4).

The surface effect which has been neglected by many writers in this field has been taken into account in a semi-empirical manner (Chapter 4). There are also given the reasons why I have not used the method developed by WEIZSACKER (W.2.) to account for the surface effect.

All the unknown constants except one are then determined numerically from data concerning the nuclei  ${}^{32}_{16}\text{S}$  and  ${}^{200}_{80}\text{Hg}$  (Chapter 6).

The results have been used to find nuclear binding energies, nuclear radii (Chapter 7), and finally leading up to a discussion of the stability of nuclei, (Chapters 8 and 9). The theoretical results are compared with the experimental results and commented on.

A BIBLIOGRAPHY has been added at the end to which reference is made in the text.



## 2. BETHE & BACHER'S STATISTICAL THEORY OF NUCLEI.

### (a). THE HARTREE METHOD.

The treatment of the nuclear statistical theory as presented in BETHE and BACHER (B.1.) will be outlined.

They use the method which has been introduced into 'atomic physics' by HARTREE (H.1.) with great success. The method consists in using the approximation that each particle is supposed to move independently of the others. They assume certain wave functions  $\psi_1, \psi_2, \dots, \psi_Z$ , for each of the  $Z$  protons and  $\varphi_1, \varphi_2, \dots, \varphi_N$ , for each of the  $N$  neutrons, each wave function being a function of the position and spin of the respective particles. Let  $p_i$  and  $n_i$  represent all coordinates of the  $i$ -th proton and neutron respectively. Then they form the following wave function for the complete nucleus:

$$(2.1.) \quad \Psi = \begin{vmatrix} \psi_1(p_1), \psi_2(p_1), \dots, \psi_Z(p_1) & \varphi_1(n_1), \varphi_2(n_1), \dots, \varphi_N(n_1) \\ \psi_1(p_2), \psi_2(p_2), \dots, \psi_Z(p_2) & \varphi_1(n_2), \varphi_2(n_2), \dots, \varphi_N(n_2) \\ \vdots & \vdots \\ \psi_1(p_Z), \psi_2(p_Z), \dots, \psi_Z(p_Z) & \varphi_1(n_N), \varphi_2(n_N), \dots, \varphi_N(n_N) \end{vmatrix}$$

This wave function evidently satisfies the demands made by PAULI'S principle.

Suppose  $V$  is the total potential energy,  $\Delta_{n_i}$  and  $\Delta_{p_i}$  are the Laplacian operators with respect to the positional coordinates of the  $i$ -th neutron and the  $i$ -th proton respectively.

Then the total energy of the nucleus is given by

$$(2.2.) \quad E = \frac{\int \Psi^* \left\{ V - (\hbar^2/2m) \left( \sum_{i=1}^N \Delta_{n_i} + \sum_{i=1}^Z \Delta_{p_i} \right) \right\} \Psi d\tau}{\int |\Psi|^2 d\tau}$$

where the integral is to be extended over all coordinates of all particles and includes summation over the spin coordinates.

This expression for the energy is actually an upper bound, giving the correct value of the energy if we insert the 'correct' wave function. Thus if it contains one or more parameters, the values of the parameters should finally be determined in such a

way as to make  $E$  a minimum. Then the approximation is as close as possible (under the assumptions made about the particular form of the wave functions) to the correct values. This method of approximation is well known as the RITZ method.

BETHE and BACHER point out a very strong argument against the validity of the HARTREE approximation in nuclear physics. One of the chief assumptions of the HARTREE theory is that the influence of the instantaneous positions of all the other particles on the motion of one particle causes only a small perturbation. This is satisfied in atomic theory, but a serious difficulty is met with in nuclear theory, due to the saturation type of nuclear forces, i.e. any given particle interacts essentially only with two particles of the other kind. Thus the force on one neutron changes by something of the order one hundred per cent according to whether a proton happens to be near the neutron or not. Thus the HARTREE method may only give a poor approximation when used in nuclear theory.

The total potential energy  $V$  is then assumed to consist of a number of terms corresponding to the interaction of pairs of particles

$$(2.3.) \quad V = \sum_{i=1}^Z \sum_{k=1}^N V_{ik}(p_i, n_k) + \frac{1}{2} \sum_{k \neq i} P_{ik}(p_i, p_k) + \frac{1}{2} \sum_{k \neq i} N_{ik}(n_i, n_k).$$

where  $V_{ik}$  represents the interaction of a proton and a neutron,  $P_{ik}$  the interaction between two protons and  $N_{ik}$  that between two neutrons.

Further they assume that  $V(p, n)$  is an interaction of the MAJORANA type i.e. if  $x$  and  $s$  denote the position and spin coordinates respectively of the proton and  $\xi$  and  $\sigma$  the corresponding values for the neutron, then

$$(2.4.) \quad V_{ik}(p_i, n_k) \psi_i(x, s) \varphi_k(\xi, \sigma) = J(x - \xi) \cdot \psi_i(\xi, s) \varphi_k(x, \sigma).$$

where  $J$  is an ordinary function of the distance between the particles.

Further they assume that each of the functions  $\psi$  and  $\varphi$  is the product of a function which depends only on the space coordinates and a function depending only on the spin coordinates.

Apart from the Coulomb interaction  $e^2/r_{12}$  they further assume that the forces between proton and neutron proton equals that between neutron and neutron and they actually take

$$(2.5.) \quad P(1, 2) = N(1, 2) = -\frac{1}{3} K(r_{12})(\vec{\sigma}_1, \vec{\sigma}_2)$$

where  $\vec{\sigma}$  denotes the PAULI spin vector and  $K(r_{12})$  is an ordinary function of the distance between the particles.

les. This means that the forces between like particles depend on the relative orientation of the spins of the particles. This type of force is a linear combination of the BARTLETT and WIGNER forces as introduced in chapter 3.

### (b). QUALITATIVE CONCLUSIONS.

Now the assumption is made that the eigenfunction corresponding to each neutron and proton is that of a free particle i.e. a plane wave. The details of the method are given in detail in chapter 4 in connection with my own development in a generalised form.

Some important conclusions can be inferred without assuming a special form of the interaction potential  $J(\tau_{12})$ . It is only assumed that;

- (1)  $J(\tau)$  vanishes at infinity sufficiently rapidly.
- (2)  $J(\tau)$  does not become infinite at  $\tau = 0$  more strongly than  $1/\tau$ .
- (3)  $J(\tau) = -Bf(\tau/a)$  where  $a$  is chosen in such a way that the main drop of the function  $f(x)$  occurs near  $x = 1$ .

The first two assumptions are necessary for the convergence of the potential energy integrals. The third assumption enables the integrals to be extended to infinity without being materially changed, and hence enabling the calculations to be made by elementary methods.

The total potential energy corresponding to the neutron-proton interaction now becomes

$$(2.6.) \quad V = -\frac{4\pi}{3} R^3 B \int_0^\infty 4\pi \tau_{12}^2 d\tau_{12} f(\tau_{12}/a) \frac{(\sin k_N \tau_{12} - k_N \tau_{12} \cos k_N \tau_{12})}{\pi^2 \tau_{12}^3} \cdot \frac{(\sin k_P \tau_{12} - k_P \tau_{12} \cos k_P \tau_{12})}{\pi^2 \tau_{12}^3}$$

where  $k_N$  and  $k_P$  are the maximum wave numbers for neutrons and protons respectively, and  $R$  stands for the radius of the nucleus. Compare with equation (4.XVI). Changing the notation by

$$\tau_0 = R(N+Z)^{1/3} : \quad \tau_{12} = \tau_0 \rho : \quad \tau_0/a = \alpha.$$

$$k_N \tau_{12} = K_N \rho : \quad k_P \tau_{12} = K_P \rho.$$

they obtain

$$(2.7.) \quad V = -\frac{16}{3} (N+Z) B \pi^{-2} \int_0^\infty f(\alpha \rho) (\sin K_N \rho - K_N \rho \cos K_N \rho) (\sin K_P \rho - K_P \rho \cos K_P \rho) \rho^{-4} d\rho.$$

$K_N$  and  $K_P$  only depend on the ratio  $N/Z$ . Actually

$$(2.8.) \quad K_N = \left[ \frac{9\pi N}{4(N+Z)} \right]^{1/3} : \quad K_P = \left[ \frac{9\pi Z}{4(N+Z)} \right]^{1/3}.$$

Thus the integral involved in  $V$  is only a function of  $N/Z$  and the parameter  $\alpha$  which determines the nuclear radius.

The kinetic energy is

$$(2.9.) \quad T = T_N + T_P = \frac{3}{5} \frac{\hbar^2}{2M} (N k_N^2 + Z k_P^2)$$

$$= \frac{3}{5} \frac{\hbar^2}{2Ma^2} \left( \frac{9\pi}{4} \right)^{2/3} (N+Z) \frac{N^{5/3} + Z^{5/3}}{(N+Z)^{5/3} \alpha^2}$$

Thus, neglecting the interaction between like particles we obtain the total energy:

$$(2.10.) \quad E = T + V = (N+Z) F(N/Z, \alpha).$$

where the function  $F$  can be explicitly determined when  $J(r)$  is known.

Determining the nuclear radius by minimizing with respect to  $\alpha$ , we see that the value of  $\alpha$  corresponding to the minimum will only depend on the ratio  $N/Z$  and not on the number of particles  $N+Z$  i.e. the nuclear radius  $R$  for a given value of  $N/Z$  is proportional to  $N+Z$ , i.e. the volume of the nucleus is proportional to the number of particles contained in it. Also we see that the binding energy  $-E$  for a given ratio of  $N$  to  $Z$  is proportional to the number of particles.

### (c). QUANTITATIVE CONCLUSIONS.

In order to calculate the binding energies and the nuclear radii, BETHE and BACHER assume

$$(2.11.) \quad J(r) = -B e^{-r^2/a^2}$$

This potential has been assumed also by FEENBERG in his calculations about light nuclei, therefore, the values of  $B$  and  $a$  are already known.

With this value of the potential they obtain:

$$(2.12.) \quad V = -B(N+Z) \pi^{-1/2} \alpha^{-3} \left\{ \left[ 2 - \alpha^2 (n^{2/3} + n^{1/3} (2-n)^{1/3} + (2-n)^{2/3}) \right] e^{-\frac{1}{4} \alpha^2 [n^{1/3} + (2-n)^{1/3}]^2} \right.$$

$$- \left[ 2 - \alpha^2 (n^{2/3} - n^{1/3} (2-n)^{1/3} + (2-n)^{2/3}) \right] e^{-\frac{1}{4} \alpha^2 [n^{1/3} + (2-n)^{1/3}]^2}$$

$$+ \pi^{1/2} \alpha^3 \Phi \left( \frac{1}{2} \alpha [n^{1/3} + (2-n)^{1/3}] \right)$$

$$\left. - \pi^{1/2} \alpha^3 \Phi \left( \frac{1}{2} \alpha [n^{1/3} - (2-n)^{1/3}] \right) \right\}$$



where

$$n = \frac{2N}{N+Z} \quad ; \quad 2-n = \frac{2Z}{N+Z} \quad ; \quad \alpha = \frac{3}{2} \left( \frac{\pi}{3} \right)^{1/3} \cdot \frac{1}{\alpha}$$

and  $\Phi(y)$  is the Gaussian error integral.

If the nucleus contains equal numbers of protons and neutrons, then  $n = 1$  and consequently

$$(2.13.) \quad V = -\pi^{1/2} \alpha^3 B (N+Z) \left\{ 2 - 3\alpha^2 + (\alpha^2 - 2)e^{-\alpha^2} + \pi^{1/2} \alpha^3 \Phi(\alpha) \right\}$$

and the kinetic energy in this case is

$$(2.14.) \quad T = \frac{3}{10} \frac{\hbar^2}{M a^2} \alpha^2 (N+Z)$$

The most satisfactory procedure now would be to insert those values of  $B$  and  $a$  derived by FEENBERG from the theory of light nuclei and then determine the minimum of  $V + T$  giving a value of  $\alpha$  which determines the nuclear radius.

The values of  $B$  and  $a$  derived from the theory of light nuclei are

$$(1) \quad B = 133 \text{ M.E.V.} \quad a = (0.99) 10^{-13} \text{ cms.}$$

if no interaction is assumed between like particles.

$$(2) \quad B = 41 \text{ M.E.V.} \quad a = (2.32) 10^{-13} \text{ cms.}$$

if an interaction between like particles is assumed. The force  $B$  in this case is the sum of the force between a proton and a neutron and half the force between like particles, because only this combination will enter into the formulae when  $n = 1$ .

The statistical model then gives the following results:

(1) Without interaction between like particles there exists no minimum of the total energy  $V + T$ , but it is positive for all values of  $\alpha$ .

(2) With interaction between like particles there exists a minimum of the energy for  $\alpha = 1.99$ . The minimum energy is then  $-2.7(N+Z)$  M.E.V.

Thus with the constants under (1) there is no binding energy, and under (2) there is a small binding energy about one quarter of the observed binding energy for medium weight nuclei. Even this small binding energy vanishes when we take into account the Coulomb repulsion and the surface energy.

BETHE and BACHER therefore conclude that the stat-

istical model is quite inadequate for the treatment of nuclear binding energies. This they attribute to the HARTREE method being unsuitable for nuclear physics as mentioned earlier in this chapter.

If we want to use the statistical method at all, we must therefore fix the constants  $B$  and  $a$  from the model itself. To fix  $B$  and  $a$ , two experimental data are necessary. Various data have been used by different writers. BETHE and BACHER take the atomic weight of  $_{80}\text{Hg}^{200}$  to be 200.016 as determined by ASTON and the nuclear radius of  $_{80}\text{Hg}^{200}$  to be  $8(10^{-13})$  cms.

They finally obtain

$$B = 70 \text{ M.E.V.} \quad a = (2.37) \cdot 10^{-13} \text{ cms.}$$

Analagous results are obtained using the simple exponential law of force

$$J(\tau) = -B e^{-\tau/a}$$

No binding energy is obtained if we insert the values of  $B$  and  $a$  derived from the theory of light nuclei. The constants necessary to give the observed binding energy and assumed nuclear radius of  $_{80}\text{Hg}^{200}$  are

$$B = 118 \text{ M.E.V.} \quad a = (1.53) \cdot 10^{-13} \text{ cms.}$$

#### (d) LIMITATIONS OF THE MODEL.

The model as adopted by BETHE and BACHER can be refined to include interactions compounded not only of MAJORANA forces but also of HEISENBERG, BARTLETT and WIGNER forces as is attempted further on.

The present model does not take into account considerations about the spin. For calculation of binding energies these extra terms are not important, but they do play an important part in the stability considerations.

From the theory of the heavy electron KEMMER (K.1.) shows that the form of the interaction potential is not  $-B e^{-\tau^2/a^2}$  or  $-B e^{-\tau/a}$  but  $-B(a/\tau) \cdot e^{-\tau/a}$ .

I have worked out similar results to the proceeding with this latest form of the interaction potential and find that by fixing the constants  $B$  and  $a$  to fit the statistical model, I obtain

$$a \approx 8 \cdot 10^{-13} \text{ cms.}$$

Now from the theory of the heavy electron FROHLICH, HEITLER and KEMMER (F.1.) have deduced the relation



(2.15.)

$$a = \frac{\hbar}{M_0 e}$$

where  $M_0$  is the mass of the heavy electron. The most reliable data on this from the present experimental work on cosmic rays gives  $M_0 = 170 m_0$ , where  $m_0$  is the mass of the electron.

Thus

(2.16.)

$$a = (2.26) 10^{-13} \text{ cms.}$$

We notice that the value of  $a$  as derived from the statistical model is too large for the potential  $-B(a/r)e^{-r/a}$  but is too small for the potential  $-Be^{-r/a}$ . This suggests that we should take the potential in the form

(2.17.)

$$-B(k + a/r)e^{-r/a}$$

i.e. a linear combination of the above potentials. The most important term for small values of  $r$  is the heavy electron potential, and for large values of  $r$  the potential is of course tending rapidly to zero. Thus I assume that the value of  $a$  is that derived from the heavy electron theory. Instead of  $a$  we now have to determine the unknown constant  $k$ . We ought to note that  $k$  is an absolute numerical constant in contrast to  $a$  which represents a length.

### 3. EXTENDED FORMULAE FOR STATISTICAL MODEL.

This chapter contains a method of treating statistically the nucleus in a more general way than that of BETHE and BACHER. The total wave function given here has been suggested by FUCHS (unpublished), who has worked out the formulae in this chapter for the different interaction energies.

The notation now begins afresh.

#### (a) NOTATION AND OPERATORS.

The total wave function of the nucleus will be represented by  $\Psi$ , whilst the spatial part of the wave function will be represented by  $\Phi$ .

Also suppose that  $S$  represents the spin function, the eigenfunctions of the spin component in the  $z$  direction being  $\alpha$  and  $\beta$ .

As in the theory of the heavy electron, we suppose that the neutron and proton are two states of one and the same particle, the eigenfunctions of the neutron and proton states being  $n$  and  $p$  respectively.

The eigenfunctions  $\alpha$  and  $\beta$  as well as  $n$  and  $p$  are respectively supposed to be normalised and orthogonal.

The argument  $i$  or the index  $i$  will refer to the  $i$ -th particle.

Operators  $P_i$  and  $N_i$  are defined by

$$(3.1.) \quad \begin{aligned} P_i n(i) &= 0 & : & & P_i p(i) &= n(i) \\ N_i n(i) &= p(i) & : & & N_i p(i) &= 0 \end{aligned}$$

From these two operators  $P_i$  and  $N_i$  we can form the exchange operator

$$(3.2.) \quad U_{ik} = P_i N_k$$

$U_{ik}$  is called an exchange operator because

$$U_{ik} p(i) n(k) = n(i) p(k)$$

Also we have

$$U_{ik} n(i) n(k) = U_{ik} n(i) p(k) = U_{ik} p(i) p(k) = 0$$

We also introduce a spin exchange operator  $S'_{ik}$  where

$$(3.3.) \quad S'_{ik} s(i, k) = s(k, i)$$

## (b). THE INTERACTION POTENTIALS.

We introduce four different types of interaction potential as follows:

$$\begin{aligned}
 (3.4.) \quad & \text{MAJORANA.} & M_{ik} &= J^M(\tau_{ik}) S'_{ik} (U_{ik} + U_{ki}). \\
 & \text{HEISENBERG.} & H_{ik} &= J^H(\tau_{ik}) (U_{ik} + U_{ki}) \\
 & \text{BARTLETT.} & B_{ik} &= J^B(\tau_{ik}) S_{ik} \\
 & \text{WIGNER.} & W_{ik} &= J^W(\tau_{ik})
 \end{aligned}$$

where the  $J(\tau_{ik})$  are ordinary functions of the distance  $\tau_{ik}$  between the  $i$ -th. and the  $k$ -th. particles.

## (c). THE TOTAL WAVE FUNCTION.

The total wave function of a nucleus consisting of  $n$  neutrons with spin  $\alpha$ ,  $m$  neutrons with spin  $\beta$ ,  $s$  protons with spin  $\alpha$ , and  $r$  protons with spin  $\beta$  is assumed in the form

$$(3.5.) \quad \Psi = \frac{1}{(A!)^{1/2}} \left| \begin{array}{c} \alpha(1)n(1)\delta'_{\nu_1}, \dots, \alpha(1)n(1)\delta'_{\nu_n}, \beta(1)n(1)\delta'_{\mu_1}, \dots, \beta(1)n(1)\delta'_{\mu_m}, \alpha(1)p(1)\delta'_{\sigma_1}, \dots, \alpha(1)p(1)\delta'_{\sigma_s}, \beta(1)p(1)\delta'_{\rho_1}, \dots, \beta(1)p(1)\delta'_{\rho_r} \\ \alpha(2)n(2)\delta'_{\nu_1}, \dots, \alpha(2)n(2)\delta'_{\nu_n}, \beta(2)n(2)\delta'_{\mu_1}, \dots, \beta(2)n(2)\delta'_{\mu_m}, \alpha(2)p(2)\delta'_{\sigma_1}, \dots, \alpha(2)p(2)\delta'_{\sigma_s}, \beta(2)p(2)\delta'_{\rho_1}, \dots, \beta(2)p(2)\delta'_{\rho_r} \\ \vdots \\ \alpha(A)n(A)\delta'_{\nu_1}, \dots, \alpha(A)n(A)\delta'_{\nu_n}, \beta(A)n(A)\delta'_{\mu_1}, \dots, \beta(A)n(A)\delta'_{\mu_m}, \alpha(A)p(A)\delta'_{\sigma_1}, \dots, \alpha(A)p(A)\delta'_{\sigma_s}, \beta(A)p(A)\delta'_{\rho_1}, \dots, \beta(A)p(A)\delta'_{\rho_r} \end{array} \right| \Phi$$

where

$$(3.6.) \quad \Phi = \Phi(\nu_1, \dots, \nu_n; \mu_1, \dots, \mu_m; \sigma_1, \dots, \sigma_s; \rho_1, \dots, \rho_r)$$

The summation convention is used for the indices  $\nu$ ,  $\mu$ ,  $\sigma$  and  $\rho$ .  $A = n + m + s + r$  represents the total number of particles in the nucleus.  $\delta_m^n$  is the usual delta operator having the value zero except when  $m = n$ , in which case its value is unity. The orthogonality and normalisation conditions are:

$$\begin{aligned}
 (3.7.) \quad & \int \Phi(\nu_1, \dots, \nu_i, \dots, \nu_j, \dots, \nu_n; \mu', \sigma', \rho) \Phi^*(\nu_1, \dots, \nu_j, \dots, \nu_i, \dots, \nu_n; \mu; \sigma; \rho) d\tau \\
 & = \delta_{ij}
 \end{aligned}$$

where  $\delta_{ij} = 1$  or  $0$  according as  $i = j$  or not. There are similar equations for the indices  $\mu, \nu$  and  $\rho$ . These conditions exclude  $\Phi$  from being symmetrical in coordinates of the same type.

$\Psi$  is obviously anti-symmetrical in all coordinates of any pair of particles and thus PAULI'S principle is satisfied.

#### (d.1.) THE MAJORANA ENERGY.

In order to calculate the interaction energies, we must apply the operators defined by (3.2.) and (3.3.) on the wave function (3.5.). We have

$$J^M(\tau_{12}) S_{12} U_{12} \Psi = \frac{1}{(A!)^2} \left| \begin{array}{c} 0, \dots, 0; 0, \dots, 0; \alpha(1)n(1)\delta_{\sigma_1}^1, \dots; \beta(2)n(2)\delta_{\rho_2}^1 \\ \alpha(1)p(1)\delta_{\nu_1}^1, \dots; \beta(1)p(2)\delta_{\mu_m}^1; 0, \dots, 0; 0, \dots, 0 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(2)n(2)\delta_{\sigma_2}^1, \dots; \beta(2)p(2)\delta_{\rho_2}^1 \\ \alpha(1)n(1)\delta_{\nu_1}^2, \dots; \beta(1)n(1)\delta_{\mu_m}^2; 0, \dots, 0; 0, \dots, 0 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(2)p(2)\delta_{\sigma_2}^2, \dots; \beta(2)p(2)\delta_{\rho_2}^2 \\ \alpha(1)p(1)\delta_{\nu_1}^2, \dots; \beta(1)p(1)\delta_{\mu_m}^2; 0, \dots, 0; 0, \dots, 0 \\ \vdots \end{array} \right| J^M(\tau_{12}) \Phi.$$

$$= \frac{1}{(A!)^2} \left| \begin{array}{c} 0, \dots, 0; 0, \dots, 0; \alpha(2)p(2)\delta_{\sigma_2}^1, \dots; \beta(2)p(2)\delta_{\rho_2}^1 \\ \alpha(1)n(1)\delta_{\nu_1}^2, \dots; \beta(1)n(1)\delta_{\mu_m}^2; 0, \dots, 0; 0, \dots, 0 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(1)n(1)\delta_{\sigma_1}^2, \dots; \beta(1)n(1)\delta_{\rho_1}^2 \\ \alpha(2)p(2)\delta_{\nu_2}^1, \dots; \beta(2)p(2)\delta_{\mu_m}^1; 0, \dots, 0; 0, \dots, 0 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(1)p(1)\delta_{\sigma_1}^1, \dots; \beta(1)p(1)\delta_{\rho_1}^1 \\ \alpha(2)n(2)\delta_{\nu_2}^2, \dots; \beta(2)n(2)\delta_{\mu_m}^2; 0, \dots, 0; 0, \dots, 0 \\ \vdots \end{array} \right| J^M(\tau_{12}) \Phi.$$

Multiplying by  $\Psi^*$  and integrating over all the coordinates of all the particles, only terms which are square in  $\alpha, \beta, n$  or  $p$  can give a non-zero contribution on account of the orthogonality conditions. Therefore, we obtain

$$\int \Psi^* J^M(\tau_{12}) S_{12} U_{12} \Psi d\tau =$$

$$\int \frac{\Phi^*}{A!} \left| \begin{array}{c} \alpha(1)n(1)\delta_{\nu_1}^1, \dots; \beta(1)n(1)\delta_{\mu_m}^1; 0, \dots, 0; 0, \dots, 0 \\ 0, \dots, 0; 0, \dots, 0; \alpha(2)p(2)\delta_{\sigma_2}^1, \dots; \beta(2)p(2)\delta_{\rho_2}^1 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(1)n(1)\delta_{\sigma_1}^2, \dots; \beta(1)n(1)\delta_{\rho_1}^2 \\ \alpha(2)p(2)\delta_{\nu_2}^1, \dots; \beta(2)p(2)\delta_{\mu_m}^1; 0, \dots, 0; 0, \dots, 0 \\ \vdots \\ 0, \dots, 0; 0, \dots, 0; \alpha(1)p(1)\delta_{\sigma_1}^1, \dots; \beta(1)p(1)\delta_{\rho_1}^1 \\ \alpha(2)n(2)\delta_{\nu_2}^2, \dots; \beta(2)n(2)\delta_{\mu_m}^2; 0, \dots, 0; 0, \dots, 0 \\ \vdots \end{array} \right| J^M(\tau_{12}) \Phi d\tau$$

i.e.

$$\int \Psi^* J^M(r_{12}) S_{12} U_{12} \Psi d\tau =$$

$$\frac{\Phi^*}{A!} n^2(1) p^2(2) \left| \begin{array}{cccc} \alpha(1) \delta_{v_1}^1 & \dots & \beta(1) \delta_{\rho_1}^1 & 0, \dots, 0, 0 \\ 0, \dots, 0, \dots, 0 & \alpha(2) \delta_{v_2}^2 & \dots & \beta(2) \delta_{\rho_2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0, \dots, 0, \dots, 0 & \alpha(1) \delta_{v_1}^2 & \dots & \beta(1) \delta_{\rho_1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0, \dots, 0, \dots, 0 & \alpha(2) \delta_{v_2}^2 & \dots & \beta(2) \delta_{\rho_2}^2 \end{array} \right| \Phi J^M(r_{12}) d\tau$$

We can expand both determinants in terms of the sub-determinants of the first two rows. Denote the sub-determinant obtained by omitting the first two rows and the  $i$ -th. and  $k$ -th. columns by sub  $\Delta_{ik}$ .

Thus the above integral has the value

$$-\frac{1}{A!} n^2(1) p^2(2) \left\{ \left[ \sum_{i=1}^n (-1)^{i-1} \alpha(1) \delta_{v_i}^1 + \sum_{j=1}^m (-1)^{n+j-1} \beta(1) \delta_{\rho_j}^1 \right] \right.$$

$$\cdot \left[ \sum_{k=1}^s (-1)^{n+m+k} \alpha(2) \delta_{v_k}^2 + \sum_{l=1}^r (-1)^{n+m+s+l} \beta(2) \delta_{\rho_l}^2 \right]$$

$$\cdot \left[ \sum_{i'=1}^n (-1)^{i'-1} \alpha(1) \delta_{v_{i'}}^2 + \sum_{j'=1}^m (-1)^{n+j'-1} \beta(1) \delta_{\rho_{j'}}^2 \right]$$

$$\cdot \left[ \sum_{k'=1}^s (-1)^{n+m+k'} \alpha(2) \delta_{v_{k'}}^1 + \sum_{l'=1}^r (-1)^{n+m+s+l'} \beta(2) \delta_{\rho_{l'}}^1 \right]$$

$$\cdot [ \text{the appropriate sub } \Delta ]$$

$$\cdot [ \text{the appropriate sub } \Delta' ] \} \Phi^* \Phi J^M(r_{12}) d\tau$$

Here we mean that each term in the product of the first four brackets is to be multiplied by the corresponding subdeterminant.

As only terms which are square in  $\alpha(1)$ ,  $\alpha(2)$ ,  $\beta(1)$  and  $\beta(2)$  can give a non zero contribution, the value of the integral is

$$-\frac{1}{A!} n^2(1) p^2(2) \left\{ \left[ \sum_{i,i'=1}^n (-1)^{i+i'} \alpha^2(1) \delta_{v_i}^1 \delta_{v_{i'}}^2 + \sum_{j,j'=1}^m (-1)^{j+j'} \beta^2(1) \delta_{\rho_j}^1 \delta_{\rho_{j'}}^2 \right] \right.$$

$$\cdot \left[ \sum_{k,k'=1}^s (-1)^{k+k'} \alpha^2(2) \delta_{v_k}^2 \delta_{v_{k'}}^1 + \sum_{l,l'=1}^r (-1)^{l+l'} \beta^2(2) \delta_{\rho_l}^2 \delta_{\rho_{l'}}^1 \right]$$

$$\cdot [ \text{the appropriate sub } \Delta ] \cdot [ \text{the appropriate sub } \Delta' ] \}$$

$$\cdot \Phi^* J^M(r_{12}) \Phi d\tau.$$



Since the product of two subdeterminants sub  $\Delta_j$  and sub  $\Delta_{k'}$  contain terms which are not square in  $\alpha, \beta$  and  $\rho$  except when  $i = i'; k = k'; j = j'; l = l'$ ; the non zero terms give a contribution:-

$$- \frac{1}{A!} \int n^2(1) \rho^2(2) \left\{ \left[ \sum_{i=1}^n \alpha^2(i) \delta_{\gamma_i}^2 \delta_{\gamma_i'}^2 + \sum_{j=1}^m \beta^2(j) \delta_{\rho_j}^2 \delta_{\rho_j'}^2 \right] \right. \\ \left. \left[ \sum_{k=1}^n \alpha^2(2) \delta_{\sigma_k}^2 \delta_{\sigma_k'}^2 + \sum_{l=1}^r \beta^2(2) \delta_{\rho_l}^2 \delta_{\rho_l'}^2 \right] \right. \\ \left. \cdot [\text{the appropriate sub } \Delta] \cdot [\text{the appropriate sub } \Delta'] \right. \\ \left. \Phi^* J^M(\tau_{12}) \Phi d\tau \right.$$

Define;

$$(3.8.) \quad f_{ik}(\gamma_i, \gamma_i'; \sigma_k, \sigma_k') \\ = \int \Phi^*(\gamma_1, \dots, \gamma_i, \dots, \gamma_n; \rho_1, \dots, \rho_m; \sigma_1, \dots, \sigma_k, \dots, \sigma_r; \rho_1, \dots, \rho_r) \\ \Phi(\gamma_1, \dots, \gamma_i', \dots, \gamma_n; \rho_1, \dots, \rho_m; \sigma_1, \dots, \sigma_k', \dots, \sigma_r; \rho_1, \dots, \rho_r) d\tau_{ik}$$

where  $d\tau_{ik}$  means integration over all the repeated coordinates i.e. over all coordinates except  $\gamma_i, \gamma_i', \sigma_k$  and  $\sigma_k'$ .

The other functions  $f_{kl}$  etc. are defined in a similar way.

Physically  $f_{ik}(\gamma_i, \gamma_i'; \sigma_k, \sigma_k')$  is the average probability of finding a particle in the  $i$ -th neutron of spin  $\alpha$  state at the point  $\gamma_i$  and a particle in the  $k$ -th proton of spin  $\alpha$  state at the point  $\sigma_k$ .

$f_{ik}(\gamma_i, \sigma_k; \sigma_k', \gamma_i')$  is the corresponding exchange probability.

Carrying out now the integration and summing over all spin and particle indices we obtain a factor  $(A-2)!$  from the subdeterminants, giving

$$\int \Phi^* J^M(\tau_{12}) S_{12} U_{12} \Phi d\tau \\ = - \frac{1}{A(A-1)} \int \left\{ \sum_{i=1}^n \sum_{k=1}^r J_{12}^M f_{ik}(1, 2; 2, 1) \right. \\ + \sum_{l=1}^n \sum_{k=1}^r J_{12}^M f_{il}(1, 2; 2, 1) \\ + \sum_{j=1}^m \sum_{k=1}^r J_{12}^M f_{jk}(1, 2; 2, 1) \\ \left. + \sum_{j=1}^m \sum_{l=1}^r J_{12}^M f_{jl}(1, 2; 2, 1) \right\} d\tau_1 d\tau_2$$



From symmetrical considerations the other  $A(A-1)-1$  operators  $S_{i\bar{k}} U_{i\bar{k}}$  and  $S_{i\bar{k}} U_{\bar{k}i}$  (1 and 2 are only dummy indices in the integral) give the same contribution.

Thus the total MAJORANA energy  $M$  is given by;

$$(3.9.) \quad M = - \int \left\{ \sum_{i=1}^n \sum_{\bar{k}=1}^S J_{i\bar{k}}^M f_{i\bar{k}}(1, 2; 2, 1) + \sum_{i=1}^n \sum_{\bar{k}=1}^T J_{i\bar{k}}^M f_{i\bar{k}}(1, 2; 2, 1) \right. \\ \left. + \sum_{j=1}^m \sum_{\bar{k}=1}^S J_{j\bar{k}}^M f_{j\bar{k}}(1, 2; 2, 1) + \sum_{j=1}^m \sum_{\bar{k}=1}^T J_{j\bar{k}}^M f_{j\bar{k}}(1, 2; 2, 1) \right\} d\tau_1 d\tau_2$$

(d.2.) THE HEISENBERG ENERGY.

For HEISENBERG forces the only difference from the previous case is that in the sums over  $i'$ ,  $j'$ ,  $\bar{k}'$  and  $\bar{l}'$  the spin indices of particles 1 and 2 are interchanged.

Thus in this case we obtain

$$\int \Psi^* J^H(\tau_{12}) U_{12} \Psi d\tau \\ = - \frac{1}{A!} \int n^2(1) p^2(2) \left[ \sum_{i=1}^n \sum_{\bar{k}=1}^S (-1)^{i+n+m+\bar{k}+i'-1} \alpha^2(1) \delta_{\nu_i}^1 \delta_{\sigma_{\bar{k}}}^1 + \sum_{j=1}^m \sum_{\bar{k}=1}^T (-1)^{j+m+s+\bar{k}+j'-1} \beta^2(1) \delta_{\mu_j}^1 \delta_{\rho_{\bar{k}}}^1 \right] \\ \cdot \left[ \sum_{\bar{k}=1}^S \sum_{i=1}^n (-1)^{n+m+\bar{k}+i'-1} \alpha^2(2) \delta_{\nu_{\bar{k}}}^2 \delta_{\nu_i}^2 + \sum_{\bar{k}=1}^T \sum_{j=1}^m (-1)^{m+s+\bar{k}+j'-1} \beta^2(2) \delta_{\rho_{\bar{k}}}^2 \delta_{\mu_j}^2 \right] \\ \cdot [\text{the appropriate sub } \Delta] \cdot [\text{the appropriate sub } \Delta'] \cdot \\ \cdot \Phi^* J_{12}^H \Phi d\tau.$$

The only terms now which give a non zero contribution are those for which  $i = i'$ ,  $\bar{k} = \bar{k}'$  and  $j = j'$ ,  $\bar{l} = \bar{l}'$ .

Thus the integral has the value;

$$- \frac{1}{A!} \int n^2(1) p^2(2) \left[ \sum_{i=1}^n \sum_{\bar{k}=1}^S \alpha^2(1) \alpha^2(2) \delta_{\nu_i}^1 \delta_{\sigma_{\bar{k}}}^2 \delta_{\nu_i}^2 \delta_{\sigma_{\bar{k}}}^1 + \sum_{j=1}^m \sum_{\bar{k}=1}^T \beta^2(1) \beta^2(2) \delta_{\mu_j}^1 \delta_{\rho_{\bar{k}}}^2 \delta_{\mu_j}^2 \delta_{\rho_{\bar{k}}}^1 \right] \\ \cdot [\text{the appropriate sub } \Delta] \cdot [\text{the appropriate sub } \Delta'] \Phi^* J_{12}^H \Phi d\tau.$$

Thus we obtain the total HEISENBERG energy  $H$  as

$$(3.10.) \quad H = - \int \left\{ \sum_{i=1}^n \sum_{\bar{k}=1}^S J_{i\bar{k}}^H f_{i\bar{k}}(1, 2; 2, 1) \right. \\ \left. + \sum_{j=1}^m \sum_{\bar{k}=1}^T J_{j\bar{k}}^H f_{j\bar{k}}(1, 2; 2, 1) \right\} d\tau_1 d\tau_2.$$

## (d.3.) THE BARTLETT &amp; WIGNER ENERGIES.

We can calculate the contributions to the energy due to BARTLETT and WIGNER forces in one calculation by taking the combined potential

$$(3.11.) \quad D_{ik} = J^p(r_{ik}) [\gamma S_{ik} - 1]$$

In order to actually find the energy we can write the total wave function  $\Psi$  in the following more convenient form where now the  $\delta$ 's also act on the particle wave functions but not on the spin functions.

$$(3.12.) \quad \Psi = \frac{1}{(A!)^{1/2}} \begin{vmatrix} \alpha(1)\delta'_{v_1}, \alpha(1)\delta'_{v_2}, \dots, \alpha(1)\delta'_{v_{n+s}}; \beta(1)\delta'_{\mu_1}, \dots, \beta(1)\delta'_{\mu_{m+r}} \\ \alpha(2)\delta^2_{v_1}, \alpha(2)\delta^2_{v_2}, \dots, \alpha(2)\delta^2_{v_{n+s}}; \beta(2)\delta^2_{\mu_1}, \dots, \beta(2)\delta^2_{\mu_{m+r}} \\ \dots \\ \alpha(A)\delta^A_{v_1}, \alpha(A)\delta^A_{v_2}, \dots, \alpha(A)\delta^A_{v_{n+s}}; \beta(A)\delta^A_{\mu_1}, \dots, \beta(A)\delta^A_{\mu_{m+r}} \end{vmatrix} \chi$$

where

$$(3.13.) \quad \chi = \Phi \cdot n(v_1) \dots n(v_n) p(v_{n+1}) \dots p(v_{n+s}) n(\mu_1) \dots n(\mu_m) p(\mu_{m+1}) \dots p(\mu_{m+r})$$

As before we can expand in terms of the subdeterminants of order  $A-2$  formed by the omission of the first two rows and any two columns of the original determinant.

We have

$$\begin{aligned} \Psi^* = & \left\{ \sum_{x < y} (-1)^{x+y-1} \alpha(1)\alpha(2) \delta'_{v_x} \delta^2_{v_y} + \sum_{z < v} (-1)^{z+v-1} \beta(1)\beta(2) \delta'_{\mu_z} \delta^2_{\mu_v} \right. \\ & + \sum_{x, z} (-1)^{x+z+n+s-1} \alpha(1)\beta(2) \delta'_{v_x} \delta^2_{\mu_z} - \text{the same expression} \\ & \left. \text{with the indices 1 and 2 interchanged} \right\} \\ & \cdot [\text{appropriate sub } \Delta] \chi^* / (A!)^{1/2}. \end{aligned}$$

Also

$$[Y S_{ik} - 1] \Psi = \left\{ \sum_{x' < y'} (\gamma - 1) (-1)^{x' + y' - 1} \alpha(1) \alpha(2) \delta_{x'}^1 \delta_{y'}^2 + \sum_{z' < v'} (\gamma - 1) (-1)^{z' + v' - 1} \beta(1) \beta(2) \delta_{z'}^1 \delta_{v'}^2 \right. \\ \left. + \sum_{x', z'} (-1)^{x' + z' + \pi + s - 1} [\gamma \alpha(2) \beta(1) - \alpha(1) \beta(2)] \delta_{x'}^1 \delta_{z'}^2 \right.$$

— the same expression with the indices 1 and 2 interchanged  $\} \cdot$

$$\cdot [\text{appropriate sub } \Delta'] \cdot \chi' / (A!)^{1/2}$$

Thus

$$\int \Psi^* J_{12}^D \Psi d\tau = \\ \frac{1}{A!} \int \left\{ \sum_{x < y} \alpha^2(1) \alpha^2(2) (\delta_{x'}^1 \delta_{y'}^2 - \delta_{x'}^2 \delta_{y'}^1) (\delta_{x''}^1 \delta_{y''}^2 - \delta_{x''}^2 \delta_{y''}^1) (\gamma - 1) \right. \\ \left. + \sum_{z < v} \beta^2(1) \beta^2(2) (\delta_{z'}^1 \delta_{v'}^2 - \delta_{z'}^2 \delta_{v'}^1) (\delta_{z''}^1 \delta_{v''}^2 - \delta_{z''}^2 \delta_{v''}^1) (\gamma - 1) \right. \\ \left. - \sum_{x, z} \alpha^2(1) \beta^2(2) \delta_{x'}^1 \delta_{z'}^2 (\delta_{x''}^1 \delta_{z''}^2 + \gamma \delta_{x''}^2 \delta_{z''}^1) \right. \\ \left. - \sum_{x, z} \alpha^2(2) \beta^2(1) \delta_{x'}^2 \delta_{z'}^1 (\delta_{x''}^2 \delta_{z''}^1 + \gamma \delta_{x''}^1 \delta_{z''}^2) \right\} \\ \cdot [\text{appropriate sub } \Delta] \cdot [\text{appropriate sub } \Delta'] \chi^* J_{12}^D \chi d\tau$$

Introduce a function  $g$  related to the wave functions  $\chi$  in exactly the same way as  $f$  was related to  $\Phi$ . Then we obtain

$$\int \Psi^* J_{12}^D \Psi d\tau \\ = \frac{1}{A(A-1)} \int \left\{ \sum_{x < y} [g_{xy}(1, 1; 2, 2) - g_{xy}(1, 2; 2, 1) \right. \\ \left. + g_{xy}(2, 2; 1, 1) - g_{xy}(2, 1; 1, 2)] (\gamma - 1) \right. \\ \left. + \sum_{z < v} [g_{zv}(1, 1; 2, 2) - g_{zv}(1, 2; 2, 1) \right. \\ \left. + g_{zv}(2, 2; 1, 1) - g_{zv}(2, 1; 1, 2)] (\gamma - 1) \right. \\ \left. - \sum_{x, z} [g_{xz}(1, 1; 2, 2) + g_{xz}(2, 2; 1, 1) \right. \\ \left. + \gamma g_{xz}(1, 2; 2, 1) + \gamma g_{xz}(2, 1; 1, 2)] \right\} J_{12}^D d\tau_1 d\tau_2$$

Since 1 and 2 are only dummy indices we can interchange them and since there are  $A(A-1)/2$  operators of the type  $\gamma_{iA-1}$ , the total energy due to both BARTLETT and WIGNER forces is

$$D = \int \left\{ \sum_{x < y} [g_{xy}(1,1;2,2) - g_{xy}(1,2;2,1)](\gamma-1) \right. \\ \left. + \sum_{z < v} [g_{zv}(1,1;2,2) - g_{zv}(1,2;2,1)](\gamma-1) \right. \\ \left. - \sum_{x,z} [g_{xz}(1,1;2,2) + \gamma g_{xz}(1,2;2,1)] \right\} J_{12}^D d\tau, d\tau_2$$

Expressing the functions  $g$  in terms of the original functions  $f$  defined by (3.8.) and returning to the former notation concerning the indices and their ranges we obtain;

$$(3.14.) \quad D = \int \left\{ \sum_{i=1}^n \sum_{i' > i}^n [f_{ii'}(1,1;2,2) - f_{ii'}(1,2;2,1)](\gamma-1) \right. \\ \left. + \sum_{A=1}^S \sum_{A' > A}^S [f_{AA'}(1,1;2,2) - f_{AA'}(1,2;2,1)](\gamma-1) \right. \\ \left. + \sum_{j=1}^m \sum_{j' > j}^m [f_{jj'}(1,1;2,2) - f_{jj'}(1,2;2,1)](\gamma-1) \right. \\ \left. + \sum_{\ell=1}^T \sum_{\ell' > \ell}^T [f_{\ell\ell'}(1,1;2,2) - f_{\ell\ell'}(1,2;2,1)](\gamma-1) \right. \\ \left. + \left[ \sum_{i=1}^n \sum_{A=1}^S f_{iA}(1,1;2,2) + \sum_{j=1}^m \sum_{\ell=1}^T f_{j\ell}(1,1;2,2) \right](\gamma-1) \right. \\ \left. - \sum_{i=1}^n \sum_{j=1}^m [f_{ij}(1,1;2,2) + \gamma f_{ij}(1,2;2,1)] \right. \\ \left. - \sum_{A=1}^S \sum_{\ell=1}^T [f_{A\ell}(1,1;2,2) + \gamma f_{A\ell}(1,2;2,1)] \right. \\ \left. - \sum_{i=1}^n \sum_{\ell=1}^T [f_{i\ell}(1,1;2,2)] - \sum_{A=1}^S \sum_{j=1}^m f_{jA}(1,1;2,2) \right\} J_{12}^D d\tau, d\tau_2.$$

#### (e). INTRODUCTION OF DENSITIES.

Densities are introduced by the following definitions:

$$(3.15.) \quad \rho_{na,p\alpha}(1,2;3,4) = \sum_{i=1}^n \sum_{A=1}^S f_{iA}(1,2;3,4) : \rho_{na,p\beta}(1,2;3,4) = \sum_{i=1}^n \sum_{\ell=1}^T f_{i\ell}(1,2;3,4) \\ \rho_{n\beta,p\alpha}(1,2;3,4) = \sum_{j=1}^m \sum_{A=1}^S f_{jA}(1,2;3,4) : \rho_{n\beta,p\beta}(1,2;3,4) = \sum_{j=1}^m \sum_{\ell=1}^T f_{j\ell}(1,2;3,4) \\ \rho_{na,n\beta}(1,2;3,4) = \sum_{i=1}^n \sum_{j=1}^m f_{ij}(1,2;3,4) : \rho_{pa,p\beta}(1,2;3,4) = \sum_{A=1}^S \sum_{\ell=1}^T f_{A\ell}(1,2;3,4) \\ \rho_{na,n\alpha}(1,2;3,4) = \sum_{i=1}^n \sum_{i' > i}^n f_{ii'}(1,2;3,4) : \rho_{n\beta,n\beta}(1,2;3,4) = \sum_{j=1}^m \sum_{j' > j}^m f_{jj'}(1,2;3,4) \\ \rho_{pa,p\alpha}(1,2;3,4) = \sum_{A=1}^S \sum_{A' > A}^S f_{AA'}(1,2;3,4) : \rho_{p\beta,p\beta}(1,2;3,4) = \sum_{\ell=1}^T \sum_{\ell' > \ell}^T f_{\ell\ell'}(1,2;3,4)$$

Physically  $\rho_{n\alpha, p\alpha}(1, 1; 2, 2)$  is the probability for a particle in any  $n\alpha$ -state at the point 1 and a particle in any  $p\alpha$ -state at the point 2. Owing to the definition of  $\rho_{n\alpha, n\alpha}(1, 1; 2, 2)$  we see that it represents only half the probability for particles in any  $n\alpha$ -state at both points 1 and 2.

$\rho_{n\alpha, p\alpha}(1, 2; 2, 1)$  and  $\rho_{n\alpha, n\alpha}(1, 2; 2, 1)$  are the corresponding exchange probabilities.

Substituting the expressions for the densities in (3.9.), (3.10.) and (3.14.) we finally obtain the various interaction energies in the form:

$$(3.16.) \quad M = - \int \{ \rho_{n\alpha, p\alpha}(1, 2; 2, 1) + \rho_{n\alpha, p\beta}(1, 2; 2, 1) + \rho_{n\beta, p\alpha}(1, 2; 2, 1) + \rho_{n\beta, p\beta}(1, 2; 2, 1) \} J_{12}^M d\tau_1 d\tau_2$$

$$(3.17.) \quad H = - \int \{ \rho_{n\alpha, p\alpha}(1, 2; 2, 1) + \rho_{n\beta, p\beta}(1, 2; 2, 1) \} J_{12}^H d\tau_1 d\tau_2$$

$$(3.18.) \quad D = \int \{ [ \rho_{n\alpha, n\alpha}(1, 1; 2, 2) - \rho_{n\alpha, n\alpha}(1, 2; 2, 1) + \rho_{n\beta, n\beta}(1, 1; 2, 2) - \rho_{n\beta, n\beta}(1, 2; 2, 1) + \rho_{p\alpha, p\alpha}(1, 1; 2, 2) - \rho_{p\alpha, p\alpha}(1, 2; 2, 1) + \rho_{p\beta, p\beta}(1, 1; 2, 2) - \rho_{p\beta, p\beta}(1, 2; 2, 1) + \rho_{n\alpha, p\alpha}(1, 1; 2, 2) + \rho_{n\beta, p\beta}(1, 1; 2, 2) ] (\gamma - 1) - [ \rho_{n\alpha, n\beta}(1, 2; 2, 1) + \rho_{p\alpha, p\beta}(1, 2; 2, 1) ] \gamma - [ \rho_{n\alpha, n\beta}(1, 1; 2, 2) + \rho_{p\alpha, p\beta}(1, 1; 2, 2) + \rho_{n\alpha, p\beta}(1, 1; 2, 2) + \rho_{n\beta, p\alpha}(1, 1; 2, 2) ] \} J_{12}^D d\tau_1 d\tau_2$$

#### 4. THE INDEPENDENT PARTICLE PICTURE.

We now introduce the assumption that each particle in the nucleus moves independently and the eigenfunction is that of a free particle i.e. a plane wave.

##### (a). PLANE WAVES.

For simplicity of the eigenfunctions we replace the spherical nucleus by a cubical box having the same volume.

Suppose  $l$  is the length of the cube,  $R$  the radius of the nucleus, and  $V$  the volume of the nucleus, so that

$$(4.1.) \quad l^3 = \frac{4\pi}{3} R^3 = V$$

The normalised plane wave function associated with the  $i$ -th. neutron of spin  $\alpha$  in such a box is

$$\frac{1}{V^{1/2}} \exp \{ i (\vec{p}_i^{n\alpha}, \vec{r}_i) \}$$

where  $\vec{r}_i$  is the position vector and  $\vec{p}_i^{n\alpha}$  is the wave vector determined from the boundary conditions.

Similarly the plane wave associated with the  $j$ -th. neutron of spin  $\beta$  is

$$\frac{1}{V^{1/2}} \exp \{ i (\vec{p}_j^{n\beta}, \vec{r}_j) \}$$

Similarly the plane waves associated with the  $k$ -th. proton with spin  $\alpha$  and the  $l$ -th. proton with spin  $\beta$  are respectively

$$\frac{1}{V^{1/2}} \exp \{ i (\vec{p}_k^{p\alpha}, \vec{r}_{\sigma k}) \}$$

and

$$\frac{1}{V^{1/2}} \exp \{ i (\vec{p}_l^{p\beta}, \vec{r}_{\sigma l}) \}$$

Thus the spatial part of the wave function is;

$$(4.2.) \quad \Phi(\gamma_1, \dots, \gamma_n; \mu_1, \dots, \mu_m; \sigma_1, \dots, \sigma_s; \rho_1, \dots, \rho_r) \\ = \frac{1}{V^{A/2}} \exp \left\{ i \sum_{i=1}^n (\vec{p}_i^{n\alpha}, \vec{r}_i) + i \sum_{j=1}^m (\vec{p}_j^{n\beta}, \vec{r}_j) \right. \\ \left. + i \sum_{k=1}^s (\vec{p}_k^{p\alpha}, \vec{r}_{\sigma k}) + i \sum_{l=1}^r (\vec{p}_l^{p\beta}, \vec{r}_{\sigma l}) \right\}$$



The components of the wave vectors  $\vec{p}^{n\alpha}$  are determined by the boundary conditions for the eigenfunctions, namely;

$$(4.3.) \quad \begin{aligned} p_x^{n\alpha} &= \frac{2\pi}{L} K_x \\ p_y^{n\alpha} &= \frac{2\pi}{L} K_y \\ p_z^{n\alpha} &= \frac{2\pi}{L} K_z \end{aligned}$$

where  $K_x, K_y, K_z$  are integers. Corresponding to each set of integers  $(K_x, K_y, K_z)$  there is one neutron state with spin  $\alpha$ , so that the total number of neutron states having wave vector components between

$$\begin{aligned} p_x^{n\alpha} \quad \text{and} \quad p_x^{n\alpha} + dp_x^{n\alpha} \\ p_y^{n\alpha} \quad \text{and} \quad p_y^{n\alpha} + dp_y^{n\alpha} \\ p_z^{n\alpha} \quad \text{and} \quad p_z^{n\alpha} + dp_z^{n\alpha} \end{aligned}$$

is

$$\left(\frac{L}{2\pi}\right)^3 dp_x^{n\alpha} dp_y^{n\alpha} dp_z^{n\alpha}$$

Thus the total number of states for which the absolute value of the wave vector is between

$$p_{n\alpha} \quad \text{and} \quad p_{n\alpha} + dp_{n\alpha}$$

is

$$\left(\frac{L}{2\pi}\right)^3 4\pi p_{n\alpha}^2 dp_{n\alpha} = \frac{2}{3\pi} R^3 p_{n\alpha}^2 dp_{n\alpha}$$

Thus the total number  $N_\alpha$  of neutron states with spin  $\alpha$  having wave numbers whose absolute value is less than  $p_{n\alpha}$  is

$$N_\alpha = \frac{2}{3\pi} R^3 \int_0^{p_{n\alpha}} p_{n\alpha}^2 dp_{n\alpha}$$

i.e.,

$$N_\alpha = \frac{2}{9\pi} R^3 p_{n\alpha}^3$$

Denoting the number of neutron states with spin  $\beta$  by  $N_\beta$  and the number of proton states with spin  $\alpha$  or  $\beta$  by  $P_\alpha$  or  $P_\beta$  respectively. We have consequently the following relations;

$$(4.4.) \quad \begin{aligned} N_\alpha &= \frac{2}{9\pi} R^3 p_{n\alpha}^3 & : & & N_\beta &= \frac{2}{9\pi} R^3 p_{n\beta}^3 \\ P_\alpha &= \frac{2}{9\pi} R^3 p_{p\alpha}^3 & : & & P_\beta &= \frac{2}{9\pi} R^3 p_{p\beta}^3 \end{aligned}$$

## (b). CALCULATION OF DENSITIES.

To calculate the densities  $\rho_{n\alpha, p\alpha}$  etc., we see from (3.8.) and (4.2.) that;

$$(4.5.) \quad \begin{aligned} f_{iR}(\gamma_i, \gamma_i'; \sigma_R, \sigma_R') \\ = \frac{1}{V^A} \int \exp \left\{ -i(\vec{p}_i^{n\alpha}, \vec{r}_{\gamma_i}) - i(\vec{p}_R^{p\alpha}, \vec{r}_{\sigma_R}) \right. \\ \left. + i(\vec{p}_i^{n\alpha}, \vec{r}_{\gamma_i'}) + i(\vec{p}_R^{p\alpha}, \vec{r}_{\sigma_R'}) \right\} d\tau_{iR} \end{aligned}$$

where  $d\tau_{iR}$  means integration over all coordinates except those of  $\gamma_i, \gamma_i'; \sigma_R, \sigma_R'$ . Thus

$$(4.6.) \quad \begin{aligned} f_{iR}(\gamma_i, \gamma_i'; \sigma_R, \sigma_R') \\ = \frac{1}{V^2} \exp \left\{ -i(\vec{p}_i^{n\alpha}, \vec{r}_{\gamma_i} - \vec{r}_{\gamma_i'}) - i(\vec{p}_R^{p\alpha}, \vec{r}_{\sigma_R} - \vec{r}_{\sigma_R'}) \right\} \end{aligned}$$

Thus

$$(4.7.) \quad f_{iR}(1, 1; 2, 2) = \frac{1}{V^2}$$

Then from the definitions (3.15.) of the densities we obtain all the direct densities as follows;

$$(4.8.) \quad \begin{aligned} \rho_{n\alpha, n\alpha}(1, 1; 2, 2) &= \frac{N_\alpha(N_\alpha-1)}{2V^2} ; \quad \rho_{n\beta, n\beta}(1, 1; 2, 2) = \frac{N_\beta(N_\beta-1)}{2V^2} ; \\ \rho_{p\alpha, p\alpha}(1, 1; 2, 2) &= \frac{P_\alpha(P_\alpha-1)}{2V^2} ; \quad \rho_{p\beta, p\beta}(1, 1; 2, 2) = \frac{P_\beta(P_\beta-1)}{2V^2} ; \\ \rho_{n\alpha, p\alpha}(1, 1; 2, 2) &= \frac{N_\alpha P_\alpha}{V^2} ; \quad \rho_{n\alpha, p\beta}(1, 1; 2, 2) = \frac{N_\alpha P_\beta}{V^2} \\ \rho_{n\beta, p\alpha}(1, 1; 2, 2) &= \frac{N_\beta P_\alpha}{V^2} ; \quad \rho_{n\beta, p\beta}(1, 1; 2, 2) = \frac{N_\beta P_\beta}{V^2} \\ \rho_{n\alpha, n\beta}(1, 1; 2, 2) &= \frac{N_\alpha N_\beta}{V^2} ; \quad \rho_{p\alpha, p\beta}(1, 1; 2, 2) = \frac{P_\alpha P_\beta}{V^2} \end{aligned}$$

To obtain the mixed densities we have from (4.6.) that

$$(4.9.) \quad f_{iR}(1,2;2,1) = \frac{1}{V^2} \exp \left\{ -i(\vec{p}_i^{na}, \vec{r}_1 - \vec{r}_2) + i(\vec{p}_R^{pa}, \vec{r}_1 - \vec{r}_2) \right\}$$

Thus from the definitions (3.15.) and from the result on page 21 that the total number of neutron states (of spin  $\alpha$ ) having wave vectors between  $\vec{p}_i^{na}$  and  $\vec{p}_i^{na} + d\vec{p}_i^{na}$  is  $\left(\frac{L}{2\pi}\right)^3 d p_x^{na} d p_y^{na} d p_z^{na}$ , we obtain

$$(4.10.) \quad \begin{aligned} \rho_{na,pa}(1,2;2,1) &= \sum_{i=1}^n \sum_{R=1}^S f_{iR}(1,2;2,1) \\ &= \frac{1}{V^2} \left(\frac{L}{2\pi}\right)^3 \left(\frac{L}{2\pi}\right)^3 \int_0^{\vec{p}_i^{na}} \int_0^{\vec{p}_R^{pa}} \left\{ \exp \left[ -i(\vec{p}_i^{na}, \vec{r}_{12}) + i(\vec{p}_R^{pa}, \vec{r}_{12}) \right] \right\} \\ &\quad d p_x^{na} d p_y^{na} d p_z^{na} d p_x^{pa} d p_y^{pa} d p_z^{pa}. \end{aligned}$$

where

$$(4.11.) \quad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

Introducing a spherical polar coordinate system in  $\vec{p}$  space, with polar axis parallel to  $\vec{r}_{12}$  we have

$$\begin{aligned} \rho_{na,pa}(1,2;2,1) &= \frac{1}{64\pi^6} \left[ \int_0^{2\pi} \int_0^\pi \int_0^{p_{na}} \sin \theta p_{na}^2 e^{-i p_{na} r_{12} \cos \theta} d p_{na} d\theta d\varphi \right] \\ &\quad \cdot \left[ \int_0^{2\pi} \int_0^\pi \int_0^{p_{pa}} \sin \theta p_{pa}^2 e^{+i p_{pa} r_{12} \cos \theta} d p_{pa} d\theta d\varphi \right] \end{aligned}$$

i.e.

$$\begin{aligned} \rho_{na,pa}(1,2;2,1) &= \frac{1}{16\pi^4} \left[ \int_0^{p_{na}} \frac{1}{i} \left[ \frac{p_{na}}{r_{12}} e^{-i p_{na} r_{12} \cos \theta} \right]_0^\pi d p_{na} \right] \\ &\quad \cdot \left[ \int_0^{p_{pa}} -\frac{1}{i} \left[ \frac{p_{pa}}{r_{12}} e^{+i p_{pa} r_{12} \cos \theta} \right]_0^\pi d p_{pa} \right] \end{aligned}$$

i.e.,

$$\begin{aligned}
 & \rho_{nd, pd}(1, 2; 2, 1) \\
 &= \frac{1}{16\pi^4} \cdot \frac{1}{r_{12}^2} \left[ \int_0^{p_{nd}} p_{nd} (e^{i p_{nd} r_{12}} - e^{-i p_{nd} r_{12}}) dp_{nd} \right] \\
 & \quad \cdot \left[ \int_0^{p_{pd}} p_{pd} (e^{-i p_{pd} r_{12}} - e^{i p_{pd} r_{12}}) dp_{pd} \right] \\
 &= \frac{1}{4\pi^4} \cdot \frac{1}{r_{12}^2} \left[ \int_0^{p_{nd}} p_{nd} \sin(p_{nd} r_{12}) dp_{nd} \right] \cdot \left[ \int_0^{p_{pd}} p_{pd} \sin(p_{pd} r_{12}) dp_{pd} \right]
 \end{aligned}$$

Thus we finally obtain

$$\begin{aligned}
 & \rho_{nd, pd}(1, 2; 2, 1) \\
 (4.12.) \quad &= \frac{1}{4\pi^4} \cdot \frac{1}{r_{12}^2} \left[ \sin(p_{nd} r_{12}) - p_{nd} r_{12} \cos(p_{nd} r_{12}) \right] \left[ \sin(p_{pd} r_{12}) - p_{pd} r_{12} \cos(p_{pd} r_{12}) \right]
 \end{aligned}$$

Similar formulae can be obtained for all the mixed densities of this type.

To obtain the other densities we have from (4.6.) that

$$\begin{aligned}
 & f_{ii'}(1, 2; 2, 1) = \frac{1}{v^2} \exp \left\{ -i (\vec{p}_{i'}^{nd}, \vec{r}_1 - \vec{r}_2) + i (\vec{p}_i^{nd}, \vec{r}_1 - \vec{r}_2) \right\} \\
 (4.13.) \quad &= \frac{1}{v^2} \exp \left\{ -i (\vec{p}_{i'}^{nd}, \vec{r}_{12}) + i (\vec{p}_i^{nd}, \vec{r}_{12}) \right\}
 \end{aligned}$$

From the definition (3.15.) we obtain

$$\begin{aligned}
 \rho_{nd, nd}(1, 2; 2, 1) &= \sum_{i=1}^n \sum_{i'=1}^n f_{ii'}(1, 2; 2, 1) \\
 &= \frac{1}{2} \left[ \sum_{i=1}^n \sum_{i'=1}^n f_{ii'}(1, 2; 2, 1) - \sum_{i=i'=1}^n f_{ii'}(1, 2; 2, 1) \right]
 \end{aligned}$$

Now from (4.13.) we have

$$[f_{ii'}(1, 2; 2, 1)]_{i=i'} = \frac{1}{v^2}$$

Thus we easily obtain

$$(4.14.) \quad \rho_{n\alpha, n\alpha}(1, 2; 2, 1) = \frac{1}{8\pi^4} \cdot \frac{1}{r_{12}^6} \cdot [\sin(p_{n\alpha} r_{12}) - p_{n\alpha} r_{12} \cos(p_{n\alpha} r_{12})]^2 - \frac{N_{\alpha}}{2v^2}$$

Similar results are obtained for all the mixed  $\alpha$  densities of this type.

(c). EXPLICIT FORM OF THE INTERACTION POTENTIAL.

As suggested in (2.17.) we make the following assumptions about the various interaction potentials; (we are now discussing the magnitude of the potentials)

$$(4.15.) \quad \begin{aligned} J^M(r_{12}) &= m \left( k + \frac{a}{r_{12}} \right) e^{-r_{12}/a} \\ J^H(r_{12}) &= h \left( k + \frac{a}{r_{12}} \right) e^{-r_{12}/a} \\ J^D(r_{12}) &= d \left( k + \frac{a}{r_{12}} \right) e^{-r_{12}/a} \end{aligned}$$

where

$$(4.16.) \quad a = (2.26) \cdot 10^{-13} \text{ cms.}$$

and  $m$ ,  $h$ ,  $d$ ,  $k$  are unknown constants which will be determined, if possible, to make the model fit some experimental data.

$m$ ,  $h$ ,  $d$ , determine the strengths of the various types of forces and have the dimensions of energy; whereas  $k$  is an absolute numerical constant.

## (d.1.). EVALUATION OF THE MAJORANA ENERGY.

The first term in the MAJORANA energy (3.16.) is

(4.XVI.)

$$- \int \rho_{n\alpha, p\alpha} (1, 2; 2, 1) J_{12}^M d\tau_1 d\tau_2$$

$$= -\frac{m}{4\pi^4} \int \left( R + \frac{a}{r_{12}} \right) \frac{e^{-r_{12}/a}}{r_{12}^6} \left[ \sin(p_{n\alpha} r_{12}) - p_{n\alpha} r_{12} \cos(p_{n\alpha} r_{12}) \right] \left[ \sin(p_{p\alpha} r_{12}) - p_{p\alpha} r_{12} \cos(p_{p\alpha} r_{12}) \right] d\tau_1 d\tau_2$$

Now the integral over  $d\tau_1 d\tau_2$  can be replaced by an integral over  $d\tau_1 d\tau_2$  i.e. first integrate over the coordinates of the particle 2 keeping the particle 1 fixed. Since the value of the integrand drops rapidly we can extend the range of integration of the relative coordinates to infinity. The integration over  $d\tau_1$  will give a factor equal to the volume of the nucleus. Integrating also over the angular portion, we obtain

$$- \int \rho_{n\alpha, p\alpha} (1, 2; 2, 1) J_{12}^M d\tau_1 d\tau_2$$

$$= -\frac{m}{4\pi^4} \cdot \frac{4\pi R^3}{3} \cdot 4\pi \int_0^\infty e^{-r_{12}/a} \left( \frac{R}{r_{12}^4} + \frac{a}{r_{12}^5} \right) \left[ \sin(p_{n\alpha} r_{12}) - p_{n\alpha} r_{12} \cos(p_{n\alpha} r_{12}) \right] \cdot \left[ \sin(p_{p\alpha} r_{12}) - p_{p\alpha} r_{12} \cos(p_{p\alpha} r_{12}) \right] \cdot d\tau_{12}.$$

Introducing a different notation by the equations:

(4.17.)

$$\chi_{n\alpha} = a p_{n\alpha}$$

$$\chi_{n\beta} = a p_{n\beta}$$

$$\chi_{p\alpha} = a p_{p\alpha}$$

$$\chi_{p\beta} = a p_{p\beta}$$

and putting

(4.18.)

$$r_{12} = a r$$

we obtain

$$- \int \rho_{n\alpha, p\alpha} (1, 2; 2, 1) J_{12}^M d\tau_1 d\tau_2$$

$$= -\frac{4}{3\pi^2} \frac{m R^3}{a^3} \int_0^\infty \left( \frac{R}{r^4} + \frac{1}{r^5} \right) e^{-r} \left[ \sin(\chi_{n\alpha} r) - \chi_{n\alpha} r \cos(\chi_{n\alpha} r) \right] \cdot \left[ \sin(\chi_{p\alpha} r) - \chi_{p\alpha} r \cos(\chi_{p\alpha} r) \right] dr$$



Introduce the functions  $\varphi(x, y)$  and  $\psi(x, y)$  defined by the equations;

$$\varphi(x, y) = \int_0^\infty \frac{e^{-\tau}}{\tau^5} [\sin(x\tau) - x\tau \cos(x\tau)] [\sin(y\tau) - y\tau \cos(y\tau)] d\tau$$

(4.19.)

$$\psi(x, y) = \int_0^\infty \frac{e^{-\tau}}{\tau^4} [\sin(x\tau) - x\tau \cos(x\tau)] [\sin(y\tau) - y\tau \cos(y\tau)] d\tau$$

Thus we obtain

$$- \int \rho_{nd, pa} (1, 2; 2, 1) J_{12}^M d\tau_1 d\tau_2$$

$$= - \frac{4}{3\pi^2} \cdot \frac{m R^3}{a^3} f(x_{nd}, x_{pa})$$

where

(4.20.)

$$f(x, y) = \varphi(x, y) + k \psi(x, y)$$

From APPENDIX A (page 87) we have the following explicit values of the functions  $\varphi(x, y)$  and  $\psi(x, y)$ ;

$$\begin{aligned} 24 \varphi(x, y) = & \left\{ \frac{1}{4} + \frac{3}{2}(x^2 + y^2) - \frac{3}{4}(x^2 - y^2)^2 \right\} \log \frac{1 + (x+y)^2}{1 + (x-y)^2} \\ & - 4(x^3 + y^3) \tan^{-1}(x+y) + 4(x^3 - y^3) \tan^{-1}(x-y) \\ & - xy + 3xy(x^2 + y^2) \end{aligned}$$

(4.21.)

$$\begin{aligned} 24 \psi(x, y) = & 4(x^3 + y^3) \tan^{-1}(x+y) - 4(x^3 - y^3) \tan^{-1}(x-y) \\ & - [1 + 3(x^2 + y^2)] \log \frac{1 + (x+y)^2}{1 + (x-y)^2} + 4xy \end{aligned}$$

Thus we finally obtain from (3.16.) that the total MAJORANA energy takes the form;

$$(4.22.) \quad M = -\frac{4}{3\pi^2} m \frac{R^3}{a^3} \left\{ f(x_{n\alpha}, x_{p\alpha}) + f(x_{n\alpha}, x_{p\beta}) \right. \\ \left. + f(x_{n\beta}, x_{p\alpha}) + f(x_{n\beta}, x_{p\beta}) \right\}$$

#### (d.2.). EVALUATION OF THE HEISENBERG ENERGY.

From equation (3.17.) we obtain immediately that the total HEISENBERG energy is

$$(4.23.) \quad H = -\frac{4}{3\pi^2} \hbar \frac{R^3}{a^3} \left\{ f(x_{n\alpha}, x_{p\alpha}) + f(x_{n\beta}, x_{p\beta}) \right\}$$

#### (d.3.). EVALUATION OF THE BARTLETT & WIGNER ENERGIES.

From equation (3.18.) giving the combined BARTLETT and WIGNER energies we have terms of the type;

$$(4.24.) \quad \int [\rho_{n\alpha, n\alpha}(1,1;2,2) - \rho_{n\alpha, n\alpha}(1,2;2,1)] J_{12}^D d\tau_1 d\tau_2 \\ = \int \left\{ \frac{N_\alpha(N_\alpha-1)}{2V^2} - \frac{1}{8\pi^4} \frac{1}{r_{12}^6} [\sin(p_{n\alpha} r_{12}) - p_{n\alpha} r_{12} \cos(p_{n\alpha} r_{12})] \right. \\ \left. + \frac{N_\alpha}{2V^2} \right\} J_{12}^D d\tau_1 d\tau_2$$

i.e.,

$$\begin{aligned}
& \int [\rho_{nd,nd}(1,1;2,2) - \rho_{nd,nd}(1,2;2,1)] J_{12}^D d\tau_1 d\tau_2 \\
&= \int \left\{ \frac{N_d^2}{2V^2} - \frac{1}{8\pi^4} \cdot \frac{1}{r_{12}^6} [\sin(p_{nd} r_{12}) - p_{nd} r_{12} \cos(p_{nd} r_{12})]^2 d\left(\frac{R}{r_{12}} + \frac{a}{r_{12}}\right) d\tau_1 d\tau_2 e^{-r_{12}/a} \right. \\
&= \frac{N_d^2 d}{2V^2} \int \left(\frac{R}{r_{12}} + \frac{a}{r_{12}}\right) e^{-r_{12}/a} d\tau_1 d\tau_2 - \frac{2d}{3\pi^2} \frac{R^3}{a^3} f(x_{nd}, x_{nd}) \\
&= \frac{2\pi}{V} N_d^2 d a^3 S - \frac{2d}{3\pi^2} \frac{R^3}{a^3} f(x_{nd}, x_{nd}) \\
&= \frac{3}{2R^3} N_d^2 d a^3 S - \frac{2d}{3\pi^2} \frac{R^3}{a^3} f(x_{nd}, x_{nd}) \\
&\text{where}
\end{aligned}$$

$$(4.24.) \quad S = \eta + R\bar{\eta}$$

$\eta$  and  $\bar{\eta}$  being defined by

$$(4.25.) \quad a^2 \eta = \int_0^R r e^{-r/a} dr.$$

$$a^3 \bar{\eta} = \int_0^R r^2 e^{-r/a} dr$$

From APPENDIX B (page 90) we have the following explicit values of the functions  $\eta$  and  $\bar{\eta}$ ;

$$(4.26.) \quad \eta = 1 - \left(1 + \frac{R}{a}\right) e^{-R/a}.$$

$$\bar{\eta} = 2 \left[ 1 - \left(1 + \frac{R}{a} + \frac{1}{2} \frac{R^2}{a^2}\right) e^{-R/a} \right].$$

We have also terms of the type;

$$\begin{aligned} & \int \rho_{n\alpha, p\alpha} (1, 1; 2, 2) J_{12}^D d\tau, d\tau_2 \\ &= \int \frac{N_\alpha P_\alpha}{V^2} d\left(k + \frac{a}{r_{12}}\right) e^{-r_{12}/a} d\tau, d\tau_2 \\ &= \frac{3}{R^3} N_\alpha P_\alpha da^3 S \end{aligned}$$

We have also terms of the type;

$$\begin{aligned} & - \int \rho_{n\alpha, n\beta} (1, 2; 2, 1) J_{12}^D d\tau, d\tau_2 \\ &= - \frac{4d}{3\pi^2} \frac{R^3}{a^3} f(x_{n\alpha}, x_{n\beta}) \end{aligned}$$

Finally we have terms of the type;

$$\begin{aligned} & - \int \rho_{n\alpha, n\beta} (1, 1; 2, 2) J_{12}^D d\tau, d\tau_2 \\ &= - \frac{3 N_\alpha N_\beta}{R^3} da^3 S \end{aligned}$$

Thus from (3.18.) we have the total energy due to both the BARTLETT and WIGNER forces is;

$$\begin{aligned} D = & -(\gamma-1) \frac{2d}{3\pi^2} \cdot \frac{R^3}{a^3} \left\{ f(x_{n\alpha}, x_{n\alpha}) + f(x_{n\beta}, x_{n\beta}) \right. \\ & \left. + f(x_{p\alpha}, x_{p\alpha}) + f(x_{p\beta}, x_{p\beta}) \right\} \\ & - \gamma \cdot \frac{4d}{3\pi^2} \cdot \frac{R^3}{a^3} \left\{ f(x_{n\alpha}, x_{n\beta}) + f(x_{p\alpha}, x_{p\beta}) \right\} \\ & + (\gamma-1) \frac{3d}{2} \frac{a^3}{R^3} S \left\{ N_\alpha^2 + N_\beta^2 + P_\alpha^2 + P_\beta^2 + 2N_\alpha P_\alpha + 2N_\beta P_\beta \right\} \\ & - \frac{3d}{2} \frac{a^3}{R^3} S \left\{ 2N_\alpha N_\beta + 2P_\alpha P_\beta + 2N_\alpha P_\beta + 2N_\beta P_\alpha \right\} \end{aligned}$$

(4.27.)

## (e). THE COULOMB FORCE.

The COULOMB force which only acts between two protons is a force of the WIGNER type. We obtain, therefore, the COULOMB energy from (3.18.) by putting

$$\gamma = 0 \quad \text{and} \quad J^p(r_{12}) = -e^2/r_{12}$$

We must take the negative sign in the potential as the COULOMB force is a repulsive force.

Thus we obtain the total COULOMB energy

$$(4.28.) \quad C = \int \left\{ \rho_{p\alpha, p\alpha}(1,1;2,2) - \rho_{p\alpha, p\alpha}(1,2;2,1) \right. \\ \left. + \rho_{p\beta, p\beta}(1,1;2,2) - \rho_{p\beta, p\beta}(1,2;2,1) \right. \\ \left. + \rho_{p\alpha, p\beta}(1,1;2,2) \right\} \frac{e^2}{r_{12}} d\eta d\tau_2$$

When evaluating the interaction energies, we replaced the spherical nucleus by a cubical box and extended the range of integration over the relative coordinate  $r_{12}$  to infinity. We were justified in doing so since the integrand drops rather quickly. In the case of the COULOMB energy the integrand will not drop off quickly and consequently we cannot extend the integration over  $r_{12}$  to infinity. As the COULOMB energy is well known, we shall not even replace the spherical nucleus by a cube, but assume the known value for a sphere.

The result is that the total COULOMB energy is:-

$$(4.29.) \quad C = \frac{3}{5} \frac{e^2}{R} (P_\alpha + P_\beta)^2 - \frac{3}{2} \frac{e^2}{R} (P_\alpha^{4/3} + P_\beta^{4/3})$$



## (f). THE KINETIC ENERGY.

The kinetic energy is obtained by the usual method of FERMI-DIRAC statistics.

If  $\rho$  is the density of particles then the kinetic energy per particle per state is

$$\frac{\hbar^2}{10\pi} \cdot \frac{1}{M} \left( \frac{3\rho}{4\pi} \right)^{5/3}$$

Thus the total kinetic energy is

$$\frac{\hbar^2}{10\pi} \frac{1}{M} \int \left( \frac{3\rho}{4\pi} \right)^{5/3} d\tau.$$

Let  $\rho_{n\alpha}$  be the density of neutrons with spin  $\alpha$ , etc. Then the total kinetic energy is

$$(4.30.) \quad T = \frac{\hbar^2}{10\pi} \cdot \frac{1}{M} \int \left\{ \left( \frac{3\rho_{n\alpha}}{4\pi} \right)^{5/3} + \left( \frac{3\rho_{n\beta}}{4\pi} \right)^{5/3} + \left( \frac{3\rho_{p\alpha}}{4\pi} \right)^{5/3} + \left( \frac{3\rho_{p\beta}}{4\pi} \right)^{5/3} \right\} d\tau$$

For plane waves we have a constant density distribution, therefore;

$$(4.31.) \quad \begin{aligned} \rho_{n\alpha} &= \frac{N_\alpha}{V} & : & \quad \rho_{n\beta} = \frac{N_\beta}{V} \\ \rho_{p\alpha} &= \frac{P_\alpha}{V} & : & \quad \rho_{p\beta} = \frac{P_\beta}{V} \end{aligned}$$

Thus we obtain the total kinetic energy of the nucleus in the form:

$$(4.32.) \quad T = \frac{3^{7/3} \hbar^2 \pi^{2/3} 2^{1/3}}{5 \cdot M \cdot 2^2} \cdot \frac{1}{R^2} [N_\alpha^{5/3} + N_\beta^{5/3} + P_\alpha^{5/3} + P_\beta^{5/3}]$$

## (g). THE SURFACE EFFECT.

In calculating the integrals occurring in the energies, we have extended the range of the limit  $R$  to infinity, neglecting in consequence the fact that particles near the surface interact with a smaller number of particles as do particles in the interior of the nucleus. Also the boundary of the nucleus cannot be sharply defined as this would mean an infinite derivative of all particle wave functions and thus the kinetic energy would become infinite.

The surface layer of the nucleus, where the density of particles drops very rapidly, must be spread out over a small region. It is obvious that if this surface layer is decreased, the kinetic energy is increased and the potential energy is increased (numerically). Thus for some intermediate surface layer there exists a minimum of the binding energy. The amount by which the binding energy is reduced by the existence of the surface layer can be called the energy due to the surface effect.

WEIZSACKER (W.2.) attempts to set up a quantitative method of estimating the value of this surface effect. He assumes that the wave functions now are no longer plane waves, but have the perturbed form;

$$\Phi = \frac{1}{V^{1/2}} [1 + (\vec{a}, \vec{r})] e^{i(\vec{p}, \vec{r})/\hbar}$$

If  $\rho$  is the density of particles he obtains that the kinetic energy density (per unit volume) is approximately

$$\frac{4\pi}{5} \frac{\rho^5}{M\hbar^3} + \frac{1}{4\pi^2\hbar M} \int \rho \vec{a}^2 d\vec{p}$$

where

$$\rho = \frac{8\pi}{3\hbar^3} \rho^3$$

Selecting  $\vec{a}$  in such a way that the energy is a minimum subject to the following condition which he derives from the definition of the density;

$$\text{grad } \rho = \frac{4}{\hbar^3} \int \rho \vec{a}(p) d\vec{p}$$

He obtains the kinetic energy in the form:

$$\int \left[ \frac{4\pi\hbar^2}{5M} \left( \frac{3\rho}{8\pi} \right)^{5/3} + \frac{\hbar^2}{32\pi^2} \frac{1}{M} \cdot \frac{(\text{grad } \rho)^2}{\rho} \right] d\tau$$

This expression for the kinetic energy can readily

be generalised to the case when there is any number of different particles.

The altered form of the potential energy due to the surface effect can also be easily obtained.

The correct form of  $\rho$  ought therefore to be deduced from the variational equation

$$\delta \int (\tau + V) d\tau = 0$$

This variational equation gives rise to a non-linear differential equation of the second order which can only be solved numerically. It would be very troublesome if not impossible to use this method here as there exist too many unknown constants.

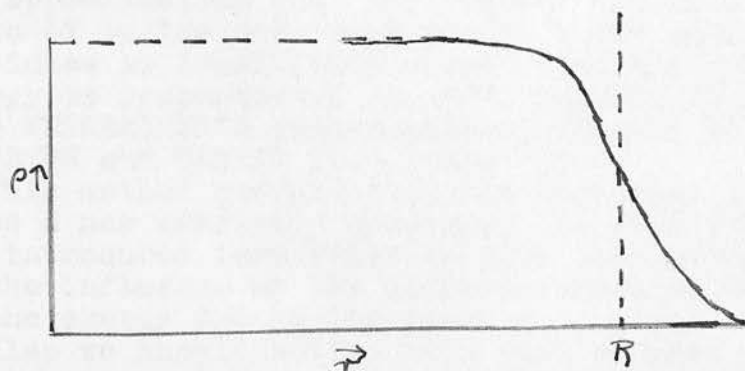
WEIZSACKER also makes no attempt to solve the variational equation. He actually now assumes that the form of the density  $\rho$  is

$$\rho = \rho_0 \left[ 1 - \frac{1}{2} e^{\lambda(\tau-R)/R} \right]^2 \quad \tau \leq R$$

$$\rho = \frac{1}{4} \rho_0 e^{-2\lambda(\tau-R)/R} \quad \tau \geq R$$

(4.33.)

This density distribution is roughly indicated in the following diagram:



The parameter  $\lambda$  introduced is then determined from a minimum condition. Then the energy due to the surface effect is completely determined.

I have attempted to estimate the surface energy in a similar manner with the density distribution (4.33.) and also with several other forms of the density distribution. I have found the kinetic energy and the potential energy (approximately) as power series in the parameter  $\lambda$ . Minimising the total binding energy with respect to  $\lambda$  and substituting back in the energy I find that the series representing the energy is not convergent. The reason probably lies in that the correct solution of the variational equation is not easily approximated to by simple analytical functions. It is of interest here to note

that WEIZSACKER did not have this difficulty. The reason is that he adopted a numerical method of solution. This is impossible here with the large number of unknown constants.

Due to these difficulties, I abandoned this method of treating the surface effect.

BETHE and BACHER (B.1.) criticize WEIZSACKER'S assumption that the decrease in density at the surface is due to a uniform decrease of the density due to each individual state. The actual fact is that as we approach the surface, one state after the other 'dies out' because its total energy becomes less than the potential energy at the given point and this dying out accounts for the decrease in the density without using an exponential decay.

As an alternative method, I have had recourse to the semi-empirical treatment i.e. a term is added to the energy involving an unknown constant (or constants) to account for the surface effect. It is known that the surface effect ought to be proportional to  $A^{2/3}$  for large nuclei. This suggests that a suitable term for the surface effect energy is

$$\frac{\omega A^{4/3}}{R^2}$$

(4.34.)

where  $\omega$  is the unknown constant introduced. With the approximation that the nuclear radius is proportional to the cube root of the total number of particles we immediately obtain that the surface energy is proportional to  $A^{2/3}$ . This is in agreement with WEIZSACKER'S semi-empirical formula as amended by BETHE and BACHER (B.1. page 165).

This method suffers from the fact that it introduces a new arbitrary constant. The form itself of the introduced term fails to give any indication of the influence of the surface effect on the portion of the energy due to the spin.

Also we should notice here that we must take one more experimental result in order to find the new constant  $\omega$ .

## (h). THE TOTAL ENERGY.

We are now in a position to write down the total energy  $E$  which consists of the sum of the MAJORANA energy, the HEISENBERG energy, the combined BARTLETT and WIGNER energies, the COULOMB energy, the KINETIC energy and the energy due to the SURFACE effect.

Changing the notation by:

$$A = \lambda^3 \quad ; \quad R = r\lambda$$

(4.35.)

$$N_\alpha = n_\alpha A : N_\beta = n_\beta A : P_\alpha = p_\alpha A : P_\beta = p_\beta A$$

From (4.4.), (4.17.) and (4.35.) we obtain

$$\chi_{n\alpha} = \left(\frac{9\pi}{2}\right)^{1/3} n_\alpha^{1/3} \cdot \frac{a}{r} : \chi_{n\beta} = \left(\frac{9\pi}{2}\right)^{1/3} n_\beta^{1/3} \cdot \frac{a}{r}$$

(4.36.)

$$\chi_{p\alpha} = \left(\frac{9\pi}{2}\right)^{1/3} p_\alpha^{1/3} \cdot \frac{a}{r} : \chi_{p\beta} = \left(\frac{9\pi}{2}\right)^{1/3} p_\beta^{1/3} \cdot \frac{a}{r}$$

Introduce the constants:

$$C_m = \frac{4m}{3\pi^2 a^3} ; C_h = \frac{4\hbar}{3\pi^2 a^3} ; C_d = \frac{2d}{3\pi^2 a^3}$$

(4.37.)

$$e_1 = \frac{3}{20} e^2 ; e_2 = \frac{3}{2^{8/3}} e^2 ; e_3 = \frac{3^{7/3} \hbar^2 \pi^{2/3}}{5 \cdot 2^{4/3} M \cdot 2^{7/3}}$$

We finally obtain the total energy in the form:

$$\begin{aligned} E = & -C_m r^3 \lambda^3 \{ f(\chi_{n\alpha}, \chi_{p\alpha}) + f(\chi_{n\alpha}, \chi_{p\beta}) + f(\chi_{n\beta}, \chi_{p\alpha}) + f(\chi_{n\beta}, \chi_{p\beta}) \} \\ & - C_h r^3 \lambda^3 \{ f(\chi_{n\alpha}, \chi_{p\alpha}) + f(\chi_{n\beta}, \chi_{p\beta}) \} \\ & - C_d (\gamma - 1) r^3 \lambda^3 \{ f(\chi_{n\alpha}, \chi_{n\alpha}) + f(\chi_{n\beta}, \chi_{n\beta}) + f(\chi_{p\alpha}, \chi_{p\alpha}) + f(\chi_{p\beta}, \chi_{p\beta}) \} \\ & - 2 C_d \gamma r^3 \lambda^3 \{ f(\chi_{n\alpha}, \chi_{n\beta}) + f(\chi_{p\alpha}, \chi_{p\beta}) \} \\ & + \frac{3 d a^3}{2 R^3} \left[ (\gamma - 1) (N_\alpha^2 + N_\beta^2 + P_\alpha^2 + P_\beta^2 + 2 N_\alpha P_\alpha + 2 N_\beta P_\beta) - (2 N_\alpha N_\beta + 2 P_\alpha P_\beta + 2 N_\alpha P_\beta + 2 N_\beta P_\alpha) \right] \\ & + \frac{4 e_1 (P_\alpha + P_\beta)^2}{R} - \frac{2^{5/3} e_2 (P_\alpha^{4/3} + P_\beta^{4/3})}{R} + \frac{\omega A^{4/3}}{R^2} \\ & + \frac{2^{4/3} e_3 [N_\alpha^{5/3} + N_\beta^{5/3} + P_\alpha^{5/3} + P_\beta^{5/3}]}{R^2} \end{aligned}$$

(4.38.)



## 5. THE EXPANSION OF THE ENERGY.

The energy (4.38.) takes the simplest form if the numbers of neutrons and protons with the two different spin values are all equal, that is;

$$(5.1.) \quad n_{\alpha} = n_{\beta} = p_{\alpha} = p_{\beta}$$

All stable nuclei lie more or less in the neighbourhood of the line given by (5.1.). Therefore we expand the energy in the neighbourhood of this line. for this purpose we introduce the variables

$$\delta = n_{\alpha} + n_{\beta} - p_{\alpha} - p_{\beta}$$

$$(5.2.) \quad s = n_{\alpha} - n_{\beta} + p_{\alpha} - p_{\beta}$$

$$t = n_{\alpha} - n_{\beta} - p_{\alpha} + p_{\beta}$$

and since obviously

$$(5.3.) \quad n_{\alpha} + n_{\beta} + p_{\alpha} + p_{\beta} = 1$$

we obtain

$$n_{\alpha} = \frac{1}{4} (1 + \delta + s + t)$$

$$(5.4.) \quad n_{\beta} = \frac{1}{4} (1 + \delta - s - t)$$

$$p_{\alpha} = \frac{1}{4} (1 - \delta + s - t)$$

$$p_{\beta} = \frac{1}{4} (1 - \delta - s + t)$$

$\delta A$  is the difference in the number of protons and neutrons, i.e. the isotopic number.

$sA/2$  is the total spin of the nucleus, the factor  $\frac{1}{2}$  arising from the fact that each particle contributes a spin  $\frac{1}{2}$ .

The variable  $t$  has no such immediate physical interpretation.

We shall proceed with the expansion of the energy by first expanding the  $\chi_{n\alpha}$ , etc. Inserting the values of  $n\alpha$ , etc., from (5.4.) into (4.36.) we obtain

$$\begin{aligned}
 (5.5.) \quad \chi_{n\alpha} &= \chi(1 + \delta v_{\alpha}) \\
 \chi_{n\beta} &= \chi(1 + \delta v_{\beta}) \\
 \chi_{p\alpha} &= \chi(1 + \delta \mu_{\alpha}) \\
 \chi_{p\beta} &= \chi(1 + \delta \mu_{\beta})
 \end{aligned}$$

where

$$(5.6.) \quad \chi = \left(\frac{9\pi}{8}\right)^{1/3} \cdot \frac{a}{r}$$

and

$$\begin{aligned}
 (5.7.) \quad \delta v_{\alpha} &= \frac{1}{3}(\delta + s + t) - \frac{1}{9}(\delta + s + t)^2 \\
 \delta v_{\beta} &= \frac{1}{3}(\delta - s - t) - \frac{1}{9}(\delta - s - t)^2 \\
 \delta \mu_{\alpha} &= \frac{1}{3}(-\delta + s - t) - \frac{1}{9}(-\delta + s - t)^2 \\
 \delta \mu_{\beta} &= \frac{1}{3}(-\delta - s + t) - \frac{1}{9}(-\delta - s + t)^2
 \end{aligned}$$

We now proceed to insert the expressions (5.5.) into the function  $f(x, y)$  introduced in (4.20.).

Then by TAYLOR'S theorem

$$\begin{aligned}
 (5.8.) \quad f[\chi(1 + \delta v), \chi(1 + \delta \mu)] &= f(x, x) + \chi(\delta v + \delta \mu) f_x \\
 &+ \chi^2 \left[ \frac{\delta v^2 + \delta \mu^2}{2} f_{xx} + \delta v \delta \mu f'_{xx} \right] \\
 &+ \dots
 \end{aligned}$$

where

$$f_x = \left[ \frac{\partial f(x, y)}{\partial x} \right]_{x=y} = \left[ \frac{\partial f(x, y)}{\partial y} \right]_{x=y}$$

(5.9.)

$$f_{xx} = \left[ \frac{\partial^2 f(x, y)}{\partial x^2} \right]_{x=y} = \left[ \frac{\partial^2 f(x, y)}{\partial y^2} \right]_{x=y}$$

$$f'_{xx} = \left[ \frac{\partial^2 f(x, y)}{\partial x \partial y} \right]_{x=y}$$

since  $f(x, y)$  is symmetrical in  $x$  and  $y$ .

Inserting the expressions (5.7.) in (5.8.) we obtain

$$f(x_{nd}, x_{nd}) = f(x, x) + \frac{2}{3} x f_x (8+s+t) + \frac{1}{9} f_1 (8+s+t)^2$$

$$f(x_{ns}, x_{ns}) = f(x, x) + \frac{2}{3} x f_x (8-s-t) + \frac{1}{9} f_1 (8-s-t)^2$$

(5.10.)

$$f(x_{pd}, x_{pd}) = f(x, x) + \frac{2}{3} x f_x (-8+s-t) + \frac{1}{9} f_1 (-8+s-t)^2$$

$$f(x_{ps}, x_{ps}) = f(x, x) + \frac{2}{3} x f_x (-8-s+t) + \frac{1}{9} f_1 (-8-s+t)^2$$

where

(5.11.)

$$f_1 = x^2 f_{xx} + x^2 f'_{xx} - 2x f_x$$

Similarly we obtain

$$\begin{aligned}
 f(x_{nd}, x_{nb}) &= f(x, x) + \frac{2x}{3} f_x \delta + \frac{1}{9} f_1 \delta^2 + \frac{1}{9} f_2 (s+t)^2 \\
 f(x_{pd}, x_{pb}) &= f(x, x) - \frac{2x}{3} f_x \delta + \frac{1}{9} f_1 \delta^2 + \frac{1}{9} f_2 (s-t)^2 \\
 f(x_{nd}, x_{pa}) &= f(x, x) + \frac{2x}{3} f_x s + \frac{1}{9} f_1 s^2 + \frac{1}{9} f_2 (\delta+t)^2 \\
 f(x_{nd}, x_{pb}) &= f(x, x) + \frac{2x}{3} f_x t + \frac{1}{9} f_1 t^2 + \frac{1}{9} f_2 (\delta+s)^2 \\
 f(x_{nb}, x_{pa}) &= f(x, x) - \frac{2x}{3} f_x t + \frac{1}{9} f_1 t^2 + \frac{1}{9} f_2 (\delta-s)^2 \\
 f(x_{nb}, x_{pb}) &= f(x, x) - \frac{2x}{3} f_x s + \frac{1}{9} f_1 s^2 + \frac{1}{9} f_2 (\delta-t)^2
 \end{aligned}
 \tag{5.12.}$$

where

$$f_2 = x^2 f_{xx} - x^2 f'_{xx} - 2x f_x$$

Forming the combinations of the  $f$ -functions which occur in the expressions for the energy, we obtain

$$\begin{aligned}
 & f(x_{na}, x_{pa}) + f(x_{nd}, x_{pb}) + f(x_{nb}, x_{pa}) + f(x_{nb}, x_{pb}) \\
 &= 4f(x, x) + \frac{2}{9} (f_1 + f_2) (s^2 + t^2) + \frac{4}{9} f_2 \delta^2 \\
 & f(x_{nd}, x_{pa}) + f(x_{nb}, x_{pb}) \\
 &= 2f(x, x) + \frac{2}{9} f_1 s^2 + \frac{2}{9} f_2 (\delta^2 + t^2) \\
 & f(x_{na}, x_{nd}) + f(x_{nb}, x_{nb}) + f(x_{pa}, x_{pa}) + f(x_{pb}, x_{pb}) \\
 &= 4f(x, x) + \frac{4}{9} f_1 (\delta^2 + s^2 + t^2) \\
 & f(x_{nd}, x_{nb}) + f(x_{pa}, x_{pb}) \\
 &= 2f(x, x) + \frac{2}{9} f_1 \delta^2 + \frac{2}{9} f_2 (s^2 + t^2)
 \end{aligned}
 \tag{5.14.}$$

For the expansion of the energy we need furthermore the expressions:

$$\begin{aligned}
 & N_\alpha^2 + N_\beta^2 + P_\alpha^2 + P_\beta^2 + 2 N_\alpha P_\alpha + 2 N_\beta P_\beta \\
 &= \lambda^6 [ (n_\alpha + p_\alpha)^2 + (n_\beta + p_\beta)^2 ] \\
 &= \frac{\lambda^6}{2} (1 + s^2) .
 \end{aligned}$$

$$\begin{aligned}
 & N_\alpha N_\beta + P_\alpha P_\beta + N_\alpha P_\beta + N_\beta P_\alpha \\
 &= \lambda^6 (n_\alpha + p_\alpha)(n_\beta + p_\beta) \\
 &= \frac{\lambda^6}{4} (1 - s^2) .
 \end{aligned}$$

$$\begin{aligned}
 & (P_\alpha + P_\beta)^2 \\
 &= \lambda^6 (p_\alpha + p_\beta)^2 \\
 (5.15.) \quad &= \frac{1}{4} \lambda^6 (1 - \delta)^2 .
 \end{aligned}$$

$$\begin{aligned}
 & P_\alpha^{4/3} + P_\beta^{4/3} \\
 &= \lambda^4 (p_\alpha^{4/3} + p_\beta^{4/3}) \\
 &= \frac{\lambda^4}{2^{5/3}} \left[ 1 - \frac{4}{3} \delta + \frac{2}{9} (\delta^2 + (s-t)^2) \right] .
 \end{aligned}$$

$$\begin{aligned}
 & N_\alpha^{5/3} + N_\beta^{5/3} + P_\alpha^{5/3} + P_\beta^{5/3} \\
 &= \lambda^5 \left[ n_\alpha^{5/3} + n_\beta^{5/3} + p_\alpha^{5/3} + p_\beta^{5/3} \right] \\
 &= \frac{\lambda^5}{2^{4/3}} \left[ 1 + \frac{5}{9} (\delta^2 + s^2 + t^2) \right] .
 \end{aligned}$$



Substituting (5.14) and (5.15.) in the expression (4.38.) for the energy, we obtain

$$\begin{aligned}
 E = & -c_m r^3 \lambda^3 \left\{ 4f(x) + \frac{2}{9}(f_1 + f_2)(s^2 + t^2) + \frac{4}{9}f_2 \delta^2 \right\} \\
 & - c_h r^3 \lambda^3 \left\{ 2f(x) + \frac{2}{9}f_1 s^2 + \frac{2}{9}f_2 (\delta^2 + t^2) \right\} \\
 & - c_d (\gamma - 1) r^3 \lambda^3 \left\{ 4f(x) + \frac{4}{9}f_1 (\delta^2 + s^2 + t^2) \right\} \\
 & - 2c_d \gamma r^3 \lambda^3 \left\{ 2f(x) + \frac{2}{9}f_1 \delta^2 + \frac{2}{9}f_2 (s^2 + t^2) \right\} \\
 & + \frac{3d}{2} \frac{a^3}{r^3} \lambda^3 \left\{ \frac{1}{2}(\gamma - 1)(1 + s^2) - \frac{1}{2}(1 - s^2) \right\} S \\
 & + \frac{e_1 \lambda^5}{r} (1 - 2\delta + \delta^2) \\
 & - \frac{e_2 \lambda^3}{r} \left\{ 1 - \frac{4}{3}\delta + \frac{2}{9}[\delta^2 + s^2 + t^2 - 2st] \right\} \\
 & + \frac{e_3 \lambda^3}{r^2} \left\{ 1 + \frac{5}{9}(\delta^2 + s^2 + t^2) \right\} \\
 & + \frac{\omega \lambda^2}{r^2} .
 \end{aligned}$$

where

$$(5.17.) \quad f(x) \equiv f(x, x) .$$

Thus we obtain the expansion of the energy in the form:

$$(5.18.) \quad E = E_0(\tau) + E_1(\tau)\delta + E_2(\tau)\delta^2 \\ + E_s(\tau)s^2 + E_t(\tau)t^2 + E_{st}(\tau)st$$

where

$$(5.19.) \quad E_0(\tau) = -f(x)\tau^3\lambda^3 \left\{ 4c_m + 2c_h + 4cd(2\gamma-1) \right\} \\ + \frac{3d}{4} \frac{a^3}{\tau^3} \lambda^3 (\gamma-2) S \\ + \frac{e_1\lambda^5 - e_2\lambda^3}{\tau} + \frac{e_3\lambda^3 + \omega\lambda^2}{\tau^2} \\ E_1(\tau) = -\frac{2e_1\lambda^5}{\tau} + \frac{4}{3} \frac{e_2\lambda^3}{\tau} \\ E_2(\tau) = -\frac{\tau^3\lambda^3}{9} \left\{ (4c_m + 2c_h)f_2 + 4cd(2\gamma-1)f_1 \right\} \\ + \frac{e_1\lambda^5}{\tau} - \frac{2}{9} \frac{e_2\lambda^3}{\tau} + \frac{5}{9} \frac{e_3\lambda^3}{\tau^2} \\ E_s(\tau) = -\frac{\tau^3\lambda^3}{9} \left\{ (2c_m + 2c_h + 4cd(\gamma-1))f_1 + (2c_m + 4cd\gamma)f_2 \right\} \\ + \frac{3d}{4} \frac{a^3}{\tau^3} \lambda^3 \gamma S - \frac{2}{9} \frac{e_2\lambda^3}{\tau} + \frac{5}{9} \frac{e_3\lambda^3}{\tau^2} \\ E_t(\tau) = -\frac{\tau^3\lambda^3}{9} \left\{ (2c_m + 4cd(\gamma-1))f_1 + (2c_h + 2c_m + 4cd\gamma)f_2 \right\} \\ - \frac{2}{9} \frac{e_2\lambda^3}{\tau} + \frac{5}{9} \frac{e_3\lambda^3}{\tau^2} \\ E_{st}(\tau) = \frac{4}{9} \frac{e_2\lambda^3}{\tau}$$

The functions  $f_1(x)$ ,  $f_2(x)$  depend only on  $\tau$ . They are calculated in APPENDIX C.

Consider now the energy  $E_0$ . This represents the actual energy in the case when there is an equal number of the four different particles. Disregarding the COULOMB energy, the surface energy and the term containing  $S$ , it can be expressed in the form

$$(5.20.) \quad E_0 = \lambda^3 F(\tau)$$

where  $F$  does not depend on the number of particles. It follows then immediately that the energy at the equilibrium radius is proportional to the number of particles. This is of course due to the saturation character of the nuclear forces. This rough approximation is well known and the slight variations can be immediately attributed to the COULOMB energy and the SURFACE effect. Consider now the term involving  $S$ .  $S$  itself depends explicitly on  $R/a$  i.e. on the total number of particles. If this term gives an appreciable contribution to the energy, it would disturb the law of proportionality of the energy to the total number of particles considerably. Therefore,  $C_d(\gamma-2)$  must be small. Consequently there are two alternatives, namely:-

- (1)  $C_d$  is small in comparison with  $C_h$  and  $C_m$ . This means that both the BARTLETT and WIGNER forces are small in comparison with the MAJORANA and HEISENBERG forces
- (2)  $\gamma$  is approximately 2. This means that the BARTLETT forces are approximately twice the WIGNER forces.

Alternative (2) is accordingly less restrictive. Therefore we shall assume that  $\gamma = 2$  exactly, so that the term involving  $S$  vanishes. Physically speaking, this means that all the nuclear forces (except the COULOMB force) shows saturation.

Some support for the assumption  $\gamma = 2$  is also obtained in a paper by VOLZ (V.1.).

It will now be more convenient to introduce instead of the force constants  $C_m$ ,  $C_h$ ,  $C_d$ , those combinations which appear in the energy. We therefore define;

$$U = 4C_m + 2C_h + 4C_d(2\gamma-1) = 4C_m + 2C_h + 12C_d$$

$$V = 4C_m + 2C_h$$

$$W_s = 2C_h - 4C_d$$

$$W_t = 2C_h + 4C_d$$

Only three of these four combinations are independent.

(5.21.)

The energy may now be written in the form (5.18.) where

$$E_0(\tau) = -u f(x) \tau^3 \lambda^3 + \frac{e_1 \lambda^5 - e_2 \lambda^3}{\tau} + \frac{e_3 \lambda^3 + \omega \lambda^2}{\tau^2}.$$

$$E_1(\tau) = -\frac{2e_1 \lambda^5}{\tau} + \frac{4}{3} \frac{e_2 \lambda^3}{\tau}.$$

$$E_2(\tau) = -\frac{\tau^3 \lambda^3}{9} \left\{ u f_1(x) - v f_3(x) \right\} + \frac{e_1 \lambda^5}{\tau} - \frac{2}{9} \frac{e_2 \lambda^3}{\tau} + \frac{5}{9} \frac{e_3 \lambda^3}{\tau^2}.$$

(5.22.)

$$E_3(\tau) = -\frac{\tau^3 \lambda^3}{18} \left\{ u f_4(x) + w f_3(x) \right\} + \frac{3d}{2} \frac{a^3}{\tau^3} \lambda^3 - \frac{2}{9} \frac{e_2 \lambda^3}{\tau} + \frac{5}{9} \frac{e_3 \lambda^3}{\tau^2}.$$

$$E_4(\tau) = -\frac{\tau^3 \lambda^3}{18} \left\{ u f_4(x) - w f_3(x) \right\} - \frac{2}{9} \frac{e_2 \lambda^3}{\tau} + \frac{5}{9} \frac{e_3 \lambda^3}{\tau^2}.$$

$$E_{st}(\tau) = \frac{4}{9} \frac{e_2 \lambda^3}{\tau}$$

where

$$f_3 = f_1 - f_2$$

$$f_4 = f_1 + f_2$$

(5.23.)

$f_3(x)$  and  $f_4(x)$  are given in APPENDIX C.

## 6. DETERMINATION OF THE CONSTANTS.

### (a). UNITS.

For the purpose of numerical calculation we introduce the following units:-

LENGTH: the electron radius =  $(2.8) \cdot 10^{-13}$  cms.

ENERGY: the mass unit. (in this unit the atomic weight of oxygen is exactly 16).

All numerical values in this and the following chapters are expressed in these units, unless stated otherwise.

Numerical calculation gives from (4.16), (5.6.) and (4.37.) that

$$a = .808$$

$$\chi = \left(\frac{9\pi}{8}\right)^{1/3} \frac{a}{r} = (1.23)/r$$

$$\epsilon_1 = \frac{3}{20} \frac{e^2}{r_{el}} = (.825) \cdot 10^{-4}$$

$$\epsilon_2 = \frac{3}{2^{8/3}} \frac{e^2}{r_{el}} = (2.598) \cdot 10^{-4}$$

$$\epsilon_3 = \frac{3^{7/3} \cdot \pi^{2/3} \cdot \hbar^2}{5 \cdot 2^3 \cdot M \cdot r_{el}^2} = (39.22) 10^{-4}$$

It should be noticed that  $r_{el}$  the radius of the electron (1 unit of length) has been introduced into the definitions of  $\epsilon_1$ ,  $\epsilon_2$  and  $\epsilon_3$  in order to make all quantities in the above units.

In the calculations we use

$$\frac{e^2}{r_{el}} = m_0 c^2 = (5.5) 10^{-4}$$

$$\frac{\hbar c}{e^2} = 137 \quad ; \quad M/m_0 = 1840$$

where  $m_0$  is the mass of the electron.



## (b). EXPERIMENTAL DATA.

We have five unknown constants  $k$ ,  $\omega$ ,  $u$ ,  $v$  and  $w_s$  or  $w_t$ . Before proceeding to apply our theoretical expressions to the determinations of various physical quantities it is desirable to obtain the values of as many of the constants as is possible. As mentioned in chapter 2 it is best to determine the constants from the model itself. In order to do this we require to assume some experimental results and apply our formulae to them.

For nuclei with  $n_\alpha = n_\beta$  and  $p_\alpha = p_\beta$ , obviously  $s = t = 0$  and therefore in this case there are only the four constants  $k$ ,  $\omega$ ,  $u$  and  $v$ . We determine these four constants from four experimental data.

At first sight it would seem desirable to select two nuclei with equal numbers of neutrons and protons, for then the energy reduces to  $E_0$  which only involves three constants, namely:  $k$ ,  $\omega$  and  $u$ . Thus we should only need three experimental data.

The largest nucleus having the property of equal numbers of neutrons and protons is  ${}_{20}\text{Ca}^{40}$ , having 20 neutrons and 20 protons. This nucleus has in all 40 particles. The second nucleus necessary for the determination of the constants must therefore have less particles, and cannot be chosen very far away from  ${}_{20}\text{Ca}^{40}$  as we cannot expect the statistical model to represent the smaller nuclei. The choice of two nuclei close together in the atomic scale gives rise to equations which are sensitive to small variations in the experimental data.

For this reason and because we wish the statistical model to fit the larger nuclei, we choose one nucleus at the lower end of the atomic scale (but not so low as to be outside the expected range of the model) and the other nucleus at the higher end of the atomic scale (but not so high as to be a radioactive nucleus).

The actual selection of the two nuclei is rather arbitrary except that we are confined to nuclei about which there are some reliable data.

I have selected the nuclei  ${}_{16}\text{S}^{32}$  and  ${}_{80}\text{Hg}^{200}$  and I use the following experimental data concerning them.

- (1) The atomic weight of  ${}_{16}\text{S}^{32}$  is 31.9823
- (2) The atomic weight of  ${}_{80}\text{Hg}^{200}$  is 200.016
- (3) The nuclear radius of  ${}_{80}\text{Hg}^{200}$  is  $(9.04)10^{-13}$  cms.
- (4) The most stable nucleus of atomic number 200 is  ${}_{80}\text{Hg}^{200}$ .

Datum (1) is taken from the table given by LIVINGSTON and BETHE (L.1. page 373).

Datum (2) is that used by BETHE (B.1. page 158) in his discussion of the statistical model.

Both these results are due to ASTON.

Datum (3) is to some extent speculative. BETHE in (B.1.) assumes the radius of  $_{80}\text{Hg}^{200}$  to be  $(8.00)10^{-13}$  cms. His justification is that the nucleus has an atomic weight just below that of the radioactive nuclei and it, therefore, seems reasonable to assume a radius slightly smaller than those of the radioactive nuclei. The table of radioactive nuclei that BETHE used is that given in (B.2. page 166) under the heading OLD. In this paper he has redetermined the radioactive nuclear radii according to the many body theory of  $\alpha$ -decay and he finds values which average about  $(3.5)10^{-13}$  cms. larger than the former values. It is probable that these nuclear radii include the radius of the helium nucleus emitted in the course of the  $\alpha$ -decay. Allowing about  $(2.5)10^{-13}$  cms. for this radius, I have decided to use a value in the neighbourhood of  $(9)10^{-13}$  cms. for the nuclear radius of  $_{80}\text{Hg}^{200}$ . (Numerical convenience made me choose 9.04 instead of 9).

Datum (4) is justified because there is only one stable nucleus with the atomic weight 200, and that nucleus is  $_{80}\text{Hg}^{200}$ . A complete table of all the known stable nuclei is given by (G.1.)

In addition it is necessary to know the atomic weights of hydrogen ( $_{1}\text{H}$ ) and the weight of the neutron. The corrected values of BETHE (B.2.) are:-  
The atomic weight of hydrogen is 1.00813.  
The weight of the neutron is 1.00897.

These values are necessary to calculate the binding energies which equals the difference between the atomic weight of the nucleus and the sum of the number of protons multiplied by the atomic weight of hydrogen ( $_{1}\text{H}$ ) and the number of neutrons multiplied by the weight of a neutron.

## (c). DETERMINATION OF THE CONSTANTS.

~~Neglecting the spin terms~~ The energy (5.18.) is given by

Note:

$$S = t = 0$$

$$(6.3.) \quad E = E_0(r) + E_1(r) \delta + E_2(r) \delta^2$$

where now from (6.1.) and (5.22.) we have

$$E_0 = -uf(x)r^3\lambda^3 + \frac{(.825)10^{-4}\lambda^5 - (.2598)10^{-4}\lambda^3}{r} + \frac{(39.22)10^{-4}\lambda^3 + \omega\lambda^2}{r^2}$$

$$(6.4.) \quad E_1(r) = -\frac{(1.65)10^{-4}\lambda^5}{r} + \frac{(3.464)10^{-4}\lambda^3}{r}$$

$$E_2(r) = -\frac{r^3\lambda^3}{9} \left\{ u f_1(x) - v f_2(x) \right\} + \frac{(.825)10^{-4}\lambda^5}{r} - \frac{(.577)10^{-4}\lambda^3}{r} + \frac{(21.789)10^{-4}\lambda^3}{r^2}$$

The radius  $r$  is of course determined by the equilibrium condition

$$(6.5.) \quad \frac{dE}{dr} = \frac{dE_0}{dr} + \frac{dE_1}{dr} \delta + \frac{dE_2}{dr} \delta^2 = 0$$

From (6.4.) we obtain for these derivatives the expressions;

$$(6.6.) \quad E'_0(r) = -ur^2\lambda^3 [3f(x) - xf'(x)] - \frac{(.825)10^{-4}\lambda^5}{r^2} + \frac{(.2598)10^{-4}\lambda^3}{r^2} - \frac{(76.44)10^{-4}\lambda^3}{r^3} - \frac{2\omega\lambda^2}{r^3}$$

$$E'_1(r) = \frac{(1.65)10^{-4}\lambda^5}{r^2} - \frac{(3.464)10^{-4}\lambda^3}{r^2}$$

$$E'_2(r) = -\frac{r^2\lambda^3}{9} \left\{ u [3f_1(x) - xf'_1(x)] - v [3f_2(x) - xf'_2(x)] \right\} - \frac{(.825)10^{-4}\lambda^5}{r^2} + \frac{(.577)10^{-4}\lambda^3}{r^2} - \frac{(43.578)10^{-4}\lambda^3}{r^3}$$

where dashes now denote derivatives with respect to  $x$ . The expressions for the various derivatives of the  $f$ -functions involved are given in APPENDIX C.

First we consider the information to be obtained from data (2) and (3) for  $^{200}_{80}\text{Hg}$ .

The total number of particles is

$$A = \lambda^3 = 200$$

Thus

$$\lambda = 5.848$$

Further

$$x = R/\lambda = .552$$

and

$$g = \frac{120-80}{200} = .2$$

Finally the binding energy is

$$\begin{aligned} -E &= -200.016 + 80[1.00813] + 120[1.00897] \\ &= 1.7108 \end{aligned}$$

Substituting these values into equation (6.3.) for the energy and into the equilibrium condition (6.5.), we obtain the equations

$$\begin{aligned} u[99.01 + 64.54k] - v[5.60 + 3.51k] \\ (6.7.) \qquad \qquad \qquad = 112.24\omega + 4.9268 \end{aligned}$$

$$\begin{aligned} u[163.51 + 56.12k] - v[9.11 + 3.85k] \\ = 224.48\omega + 5.8472 \end{aligned}$$

From datum (4) we obtain for  $^{200}_{80}\text{Hg}$  the stability condition

$$(6.8.) \quad \frac{\partial E}{\partial \delta} \equiv E_1 + 2E_2\delta = 0 \quad \text{for } \delta = -2.$$

i.e., on substitution

$$(6.9.) \quad \mu [39.02 + 9.68k] - \nu [139.97 + 87.73k] = -2.3662$$

From these three equations (6.7.), (6.8.) and (6.9.) we obtain the following three equations

$$(A) \quad \mu = \frac{565.75 + 358.99k}{4749.8 + 13097k + 6370.3k^2}$$

$$(B) \quad \nu = \frac{238.02 + 211.44k}{4749.8 + 13097k + 6370.3k^2}$$

$$(C) \quad \omega = \frac{278.75 + 49.13k - 79.82k^2}{4749.8 + 13097k + 6370.3k^2}$$

Finally we obtain the remaining information from datum (1) for  $^{32}_{16}\text{S}$ . This nucleus has an equal number of protons and neutrons, giving  $\delta = 0$ . The spin is zero giving  $S = 0$ . As with  $^{200}_{80}\text{Hg}$  we take  $t = 0$ .

The total number of particles is

$$A = \lambda^3 = 32$$

Thus

$$\lambda = 3.175$$

Finally the binding energy is

$$\begin{aligned} -E &= -31.9823 + 16[1.00813 + 1.00897] \\ &= 2913 \end{aligned}$$





Substituting these values in (6.3.) and (6.5.) we obtain the equations

$$(6.10.) \quad -.2913 = -32 u r^3 f(x) + \frac{1.83}{r} 10^{-2} + \frac{1.2552}{r^2} 10^{-1} + \frac{10.08}{r^2} \omega$$

$$0 = -32 u r^2 [3f(x) - x f'(x)] - \frac{1.83}{r^2} 10^{-2} - \frac{2.5104}{r^3} 10^{-1} - \frac{20.16}{r^3} \omega$$

We combine these two equations and use the following two equations derived therefrom:

$$(D) \quad 32 u r^3 [5f(x) - x f'(x)] = .5826 + \frac{1.83}{r} 10^{-2}.$$

$$(E) \quad \frac{10.08 \omega}{r^2} = 32 u r^3 f(x) - \frac{1.83}{r} 10^{-2} - \frac{1.2552}{r^2} 10^{-1} - .2913.$$

In conjunction with these we have equation (6.1.), namely

$$(6.11.) \quad x r = 1.23$$

We solve the equations (A), (B), (C), (D) AND (E) numerically. The method of solution is as follows:-

Assume a value of  $r$  (preferably a value suggested by a very rough solution of the equations). Substitute this value of  $r$  in the five equations (A), (B), (C), (D), (E), having obtained  $x$  from equation (6.11.). Eliminating  $u$  from equations (A) AND (D) we obtain a quadratic equation for  $k$  which can easily be solved. (We must choose the positive root of the quadratic equation because a negative value of  $k$  would make the various forces attractive or repulsive according to their mutual distance apart.) Now substituting for  $k$  in the equations (A), (B) and (C) we obtain the values of  $u$ ,  $v$  and  $\omega$ . Substituting for  $r$ ,  $u$  and  $k$  in equation (E) we obtain in general a value of  $\omega$  different from that value of  $\omega$  derived from equation (C).

For the correct value of  $r$  the two solutions for  $\omega$  must be identical. By trying various values



of  $\chi$ , I find the following solution of the equations;

$$\begin{aligned}
 (6.12.) \quad u &= (2.85) 10^{-2} \\
 v &= (1.43) 10^{-2} \\
 w &= (4.45) 10^{-3} \\
 r &= .566 \\
 k &= 1.5
 \end{aligned}$$

Then from (5.21.) we obtain

$$\begin{aligned}
 (6.13.) \quad 2C_m + C_k &= (7.15) \cdot 10^{-3} \\
 C_d &= (1.18) \cdot 10^{-3}
 \end{aligned}$$

Then from (4.37.) we obtain

$$\begin{aligned}
 (6.14.) \quad 2m + k &= (2.79) \cdot 10^{-2} \\
 d &= (0.92) \cdot 10^{-2}
 \end{aligned}$$

In order to completely determine the constants, we still require to obtain the numerical value of one other combination of the force constants involving  $w_s$  or  $w_t$ . From (5.22.) we see that  $w_s$  and  $w_t$  only occur in the spin terms of the energy. For medium and large nuclei the values of  $s$  and  $t$  are small in comparison with  $\delta$  and numerical calculation shows that the contribution of the spin terms to the energy is negligible. It is, therefore, not practicable to determine the last unknown constant from energy considerations.

It is interesting to compare the values of  $2m+h$  obtained in this statistical method with the corresponding values derived by BETHE and BACHER (B.I.) for the deuteron. (As the deuteron is composed of a proton and a neutron the BARTLETT and WIGNER forces drop out).

Denoting by large letters the corresponding quantities in the theory of the deuteron, we have

$$M = 30 \text{ M.E.V.} \quad H = 5 \text{ M.E.V.}$$

Thus

$$\begin{aligned} 2M + H &= 65 \text{ M.E.V.} \\ &= (7.00) 10^{-2} \text{ (mass units)}. \end{aligned}$$

Thus

$$\frac{2M + H}{2m + h} = 2.51$$

Owing to the fact that the potential is now proportional to  $(k + a/r)\exp(-r/a)$ , the above ratio should not be unity but rather a value approximately equal to  $k + a/r$  when  $r$  is the mean distance between the interacting particles. This value of  $r$  is at most the value of  $r$  for the nucleus.

From graph I we see that the average value of  $r$  for the various nuclei (leaving out the smaller nuclei) is about .55.

From (6.1.) we have  $a = .808$ . Thus

$$k + \frac{a}{r} = 2.97$$

Thus the true relation between  $2m+h$  and  $2M+H$  should be

$$(2m+h) < .34(2M+H)$$

whilst the statistical model gives the relation

$$2m + h = .40(2M + H)$$

Thus the forces are at least about twenty per cent too large. This is in full agreement with the following statement of BETHE and BACHER (B.I.) page 158 that:- 'Any expression for the nuclear forces which would make the nuclear binding energy following from the statistical model equal to the observed binding energy must certainly give too large values for the nuclear forces.'

Assuming that the ratio  $c_m / c_h$  is equal in both cases, we obtain

$$\frac{c_m}{c_h} = 6 = \frac{m}{h}$$

Thus

$$m = (1.29) 10^{-2}$$

$$h = (0.21) 10^{-2}$$

Also we now have

$$C_m = (3.30) \cdot 10^{-3}$$

$$(6.15.) \quad C_h = (0.55) \cdot 10^{-3}$$

$$w_s = - (3.62) \cdot 10^{-3}$$

$$w_t = (5.82) \cdot 10^{-3}$$

As  $C_h$  has only in any case a small influence on the spin terms of the energy, the above assumption giving  $C_h$  does not influence the calculations later on concerning the stability of nuclei to any appreciable extent.

## 7. NUCLEAR ENERGIES & RADII.

With the values (6.12.) of the constants determined in the preceeding chapter, we can calculate the radius and energy of any nucleus. We have to determine its radius from the equilibrium condition (6.5.), namely:-

$$(7.1.) \quad E_0'(\tau) + E_1'(\tau)\delta + E_2'(\tau)\delta^2 + E_3'(\tau)s^2 + E_4'(\tau)t^2 + E_{st}'(\tau)st = 0$$

where we now include the terms due to the spin.

Inserting the solution  $\tau$  of this equation in the expression (5.18.) we obtain the energy.

This method is not very adaptable to numerical computation. Instead we proceed in the following manner.

First we develop the radius  $\tau$  in terms of  $\delta$ ,  $s$  and  $t$ . The development is

$$(7.2.) \quad \tau = \tau_0 + \tau_1\delta + \tau_2\delta^2 + \tau_3s^2 + \tau_4t^2 + \tau_{st}st$$

Substituting this expression of  $\tau$  into the equilibrium condition (7.1.) and expanding each term using TAYLOR'S theorem, we obtain

$$\begin{aligned} & E_0'(\tau_0) + (\tau_1\delta + \tau_2\delta^2 + \tau_3s^2 + \tau_4t^2 + \tau_{st}st) E_0''(\tau_0) \\ & + \frac{1}{2} (\tau_1\delta + \tau_2\delta^2 + \tau_3s^2 + \tau_4t^2 + \tau_{st}st)^2 E_0'''(\tau_0) \\ & + \delta \{ E_1'(\tau_0) + (\tau_1\delta + \tau_2\delta^2 + \tau_3s^2 + \tau_4t^2 + \tau_{st}st) E_1''(\tau_0) \} \\ & + \delta^2 E_2'(\tau_0) + s^2 E_3'(\tau_0) + t^2 E_4'(\tau_0) \\ & + st E_{st}'(\tau_0) = 0. \end{aligned}$$

Arranging the terms as a power series in  $\delta$ ,  $s$  and  $t$ , and neglecting terms of higher order than the second we obtain

$$\begin{aligned}
 (7.3.) \quad & E_0'(\tau_0) + \delta [E_1'(\tau_0) + \tau_1 E_0''(\tau_0)] \\
 & + \delta^2 [E_2'(\tau_0) + \tau_1 E_1''(\tau_0) + \frac{1}{2} \tau_1^2 E_0'''(\tau_0) + \tau_2 E_0''(\tau_0)] \\
 & + s^2 [E_s'(\tau_0) + \tau_s E_0''(\tau_0)] + t^2 [E_t'(\tau_0) + \tau_t E_0''(\tau_0)] \\
 & + st [E_{st}'(\tau_0) + \tau_{st} E_0''(\tau_0)] = 0
 \end{aligned}$$

This condition is to be true for all possible values of  $\delta$ ,  $s$  and  $t$ , thus we obtain the relations

$$\begin{aligned}
 (7.4.) \quad & E_0'(\tau_0) = 0 \\
 & E_1'(\tau_0) + \tau_1 E_0''(\tau_0) = 0 \\
 & E_2'(\tau_0) + \tau_1 E_1''(\tau_0) + \frac{1}{2} \tau_1^2 E_0'''(\tau_0) + \tau_2 E_0''(\tau_0) = 0 \\
 & E_s'(\tau_0) + \tau_s E_0''(\tau_0) = 0 \\
 & E_t'(\tau_0) + \tau_t E_0''(\tau_0) = 0 \\
 & E_{st}'(\tau_0) + \tau_{st} E_0''(\tau_0) = 0
 \end{aligned}$$

Solving for the various  $\tau$ 's, we obtain

$$\begin{aligned}
 (7.5.) \quad & \tau_1 = - \frac{E_1'(\tau_0)}{E_0''(\tau_0)} \\
 & \tau_2 = - \frac{E_2'(\tau_0)}{E_0''(\tau_0)} + \frac{E_1'(\tau_0) E_1''(\tau_0)}{[E_0''(\tau_0)]^2} - \frac{1}{2} \frac{E_0'''(\tau_0) [E_1'(\tau_0)]^2}{[E_0''(\tau_0)]^3} \\
 & \tau_s = - \frac{E_s'(\tau_0)}{E_0''(\tau_0)} \\
 & \tau_t = - \frac{E_t'(\tau_0)}{E_0''(\tau_0)} \\
 & \tau_{st} = - \frac{E_{st}'(\tau_0)}{E_0''(\tau_0)}
 \end{aligned}$$

Substituting in the expression (5.18.) of the energy the expression (7.2.) for  $r$ , we obtain on expansion by TAYLOR'S theorem

$$\begin{aligned}
 (7.9.) \quad E = & E_0(r_0) + (r_1\delta + r_2\delta^2 + r_3s^2 + r_4t^2 + r_{st}st) E_0'(r_0) \\
 & + \frac{1}{2} (r_1\delta + r_2\delta^2 + r_3s^2 + r_4t^2 + r_{st}st)^2 E_0''(r_0) \\
 & + \delta \{ E_1(r_0) + (r_1\delta + r_2\delta^2 + r_3s^2 + r_4t^2 + r_{st}st) E_1'(r_0) \} \\
 & + E_2(r_0) \delta^2 + E_3(r_0) s^2 + E_4(r_0) t^2 \\
 & + E_{st}(r_0) st
 \end{aligned}$$

Retaining terms up to and including the second order in  $\delta$ ,  $s$  and  $t$  after substituting for  $r_1$ ,  $r_2$  their values from (7.5.) we obtain

$$\begin{aligned}
 (7.6.) \quad E = & E_0(r_0) + \delta E_1(r_0) + \delta^2 \bar{E}_2(r_0) \\
 & + s^2 E_3(r_0) + t^2 E_4(r_0) + st E_{st}(r_0)
 \end{aligned}$$

where

$$(7.7.) \quad \bar{E}_2(r_0) = E_2(r_0) - \frac{1}{2} \frac{[E_1'(r_0)]^2}{E_0''(r_0)}$$

Also substituting in (7.2.) the values of the various  $r$ 's from (7.5.), we obtain for the radius the expression

$$\begin{aligned}
 (7.8.) \quad r = & r_0 - \delta \frac{E_1'(r_0)}{E_0''(r_0)} \\
 & + \delta^2 \left\{ -\frac{E_2'(r_0)}{E_0''(r_0)} + \frac{E_1'(r_0) E_1''(r_0)}{[E_0''(r_0)]^2} - \frac{1}{2} \frac{E_0'''(r_0) [E_1'(r_0)]^2}{[E_0''(r_0)]^3} \right\} \\
 & - s^2 \frac{E_3'(r_0)}{E_0''(r_0)} - t^2 \frac{E_4'(r_0)}{E_0''(r_0)} - st \frac{E_{st}'(r_0)}{E_0''(r_0)}
 \end{aligned}$$



Thus we have expressed the energy and radius in terms of  $r_0$ , and  $\delta$ ,  $s$  and  $t$ .

$r_0$  is the solution of the equation (7.4.)

(7.9.)

$$E_0'(r_0) = 0$$

The advantage now is that (7.9.) is a simpler equation to treat than (7.1.). Also  $r_0$  is independent of  $\delta$ ,  $s$  and  $t$ .

The numerical method of procedure now is to solve (7.9.) for any nucleus. By substitution in (7.8.) and (7.6.) we obtain the values of the radius and the energy.

The equation (7.9.) is solved numerically and the calculations have been carried out with a possible error in the third place of decimals in the final results.

To some particular values of  $A$  there corresponds sometimes two values of  $\delta$  giving rise to stable nuclei. In general the choice of  $\delta$  does not affect the radius or energy sensitively. (We may note here that  $\delta A$  can only differ by 2 when  $A$  is odd and by 4 when  $A$  is even for stable nuclei corresponding to a given value of  $A$ . There exist the two exceptions for  $A = 102$  and  $124$  when there exist three stable nuclei, in which case  $\delta A$  may differ by 8. These two exceptions need not concern us here). I have in fact chosen  $\delta$  arbitrarily.

In the calculations I have neglected the spin terms. This is justified because  $s$  and  $t$  are small in comparison with  $\delta$  (except for the smaller nuclei) making in consequence the spin terms negligible.

It is of interest to note that in these calculations we do not require the values of all the three force constants, but only the two linear combinations  $u$  and  $v$ . The third combination

$w_s$  or  $w_t$  only occurs in the neglected spin terms.

Graphs of  $r$ ,  $R$ ,  $E$  and the binding energy per particle have been drawn against the total number of particles  $A$ .

## DISCUSSION OF THE VARIOUS GRAPHS.

Graph I (page 81) of  $r$  against  $A$  shows a rapid decrease of  $r$  as  $A$  increases and then slowly reaches a minimum for  $A$  about 130. After passing the minimum  $r$  begins to increase slowly with increasing  $A$ .

This is due to the opposing effects of the COULOMB energy and the surface effect. As pointed out in chapter 5 page 44,  $r$  is a constant if we neglect these two energies. Both the COULOMB energy and the surface energy are positive and thus tend to decrease the binding energy and hence lead to a larger radius. For small nuclei, the surface effect becomes more and more important, leading to the rise of  $r$  with decreasing  $A$ . For large nuclei, on the other hand, the COULOMB energy becomes more and more important, leading to the rise of  $r$  with increasing  $A$ .

Graph II (page 82) of  $R$  against  $A$  shows the steady increase of the actual nuclear radius  $R$  with  $A$ . There is a point of inflection at about  $A \sim 130$  corresponding to the minimum of  $r$  there in graph I.

Graph III (page 83) of the energy  $E$  against  $A$  shows the almost linear increase of the energy with the atomic number  $A$ . The crosses denote some experimental values of the energy.

The theoretical and experimental values of the energy have been made to fit at  $A = 32$  and  $A = 200$ . It is not very remarkable, therefore, that the theoretical energy curve fits very closely to the experimental values in the range (32, 200). However it is remarkable that the theoretical values of the energy agree well with the experimental values as far down as  $A = 14$ .

Graph IV (page 84) of the binding energy per particle  $-E/A$  against the total number of particles  $A$  is more interesting. This graph, of course, shows the same good agreement with experimental results as graph III.

## 8. APPLICATION OF THE STATISTICAL MODEL TO STABILITY.

Now we are going to apply the statistical model as developed above to the stability of nuclei. That is, corresponding to any given number of nuclear particles, we determine the composition of the stable nucleus.

### (a). COMPOSITION OF THE NUCLEUS.

For the purpose of this chapter it will be more convenient to introduce  $\Delta$ ,  $S'$  and  $T$  instead of  $\delta$ ,  $s$  and  $t$ , where

$$\begin{aligned}\Delta &= A\delta \\ S &= As \\ T &= At\end{aligned}$$

(8.1.)

For any nucleus  $\Delta$ ,  $S'$  and  $T$  are integers. In fact, from (8.1.) and (5.2.) and (4.35.) we obtain

$$\begin{aligned}\Delta &= N_\alpha + N_\beta - P_\alpha - P_\beta \\ S' &= N_\alpha - N_\beta + P_\alpha - P_\beta \\ T &= N_\alpha - N_\beta - P_\alpha + P_\beta\end{aligned}$$

(8.2.)

The composition of a nucleus is determined if the total number of particles  $A$  and the 'ISOTOPIC NUMBER'  $\Delta$  is given.

$S'$  and  $T$  define different quantum states of the nucleus (i.e. different spins). There are in addition also excited quantum states.

We shall denote by

$$E(A, \Delta, S', T)$$

(8.3.)

the energy of the nucleus determined by  $A$ ,  $\Delta$ ,  $S'$  and  $T$ .

$E$  is used throughout to represent the energy. The binding energy is of course  $-E$ .

## (b). STABILITY CONDITIONS.

Disregarding (as we shall do)  $\alpha$ -disintegration, there are three processes by which a nucleus may disintegrate:-

(1.) The nucleus emits an electron.

(2.) The nucleus emits a positron.

(3.) The nucleus captures an electron.

We need not be concerned with the possibility of a nucleus becoming unstable by capturing a positron. Electrons, on the other hand, can be captured from the K-shell of the atom and thus capture of an electron must be considered in the discussion of stability.

Considering case (1.) of  $\beta^-$  disintegration, the produced nucleus has one less neutron and one more proton. The isotopic number of the produced nucleus is therefore  $\Delta - 2$  where  $\Delta$  is the corresponding value for the original nucleus. If  $S'$  and  $T'$  are the new values of  $S$  and  $T$ , the produced nucleus has the energy

$$E(A, \Delta - 2, S', T')$$

Denoting the rest mass of the electron by  $\mu$ , we have the condition of stability

$$(8.4.) \quad E(A, \Delta, S, T) < E(A, \Delta - 2, S', T') + \mu$$

This stability condition is given by FUCHS (F.2.), equation (9). The equation only has a slightly different form there because FUCHS denotes binding energies by  $E$ . Expressed in words by BETHE and BACHER (B.1.p101), the condition states:-

'A nucleus is unstable against  $\beta^-$ -disintegration, if the replacement of a neutron in the nucleus by a proton would make the energy of the corresponding atom smaller.'

Considering case (2) of  $\beta^+$ -disintegration, the produced nucleus has one more neutron and one less proton. The isotopic number of the produced nucleus is therefore  $\Delta + 2$ . The produced nucleus now has the energy represented by

$$E(A, \Delta + 2, S', T')$$

Taking the rest mass of the positron to be the same as that of the electron, we obtain the stability condition

$$(8.5.) \quad E(A, \Delta, S, T) < E(A, \Delta+2, S', T') + \mu$$

Expressed in words this condition states:-  
'A nucleus is unstable against  $\beta^+$ -disintegration, if the replacement of a proton in the nucleus by a neutron would make the energy of the corresponding atom smaller!'

Considering case (3) of electron absorption, the produced nucleus has the isotopic number  $\Delta+2$ .

The stability condition now is

$$(8.6.) \quad E(A, \Delta, S, T) + \mu < E(A, \Delta+2, S', T')$$

We see that condition (8.5.) is contained in (8.6.) and thus there are only the two conditions (8.4.) and (8.6.) of stability.

### (c). THE NUCLEAR SPIN.

For a given nucleus (i.e. a nucleus for which  $A$  and  $\Delta$  are given)  $S$  and  $T$  are not yet completely determined, but give states with various spins. We are interested in the lowest state and consequently we have to choose  $S$  and  $T$  in such a way that the energy due to the spin terms is a minimum.

From (7.6.) and (8.1.) the energy due to the spin terms is

$$(8.7.) \quad \frac{1}{A^2} \left\{ S^2 E_s(r_0) + T^2 E_t(r_0) + S' T E_{st}(r_0) \right\}$$

From (5.22.) we see that  $E_{st}$  only arises from the COULOMB exchange energy which is very small. Thus the term in  $S'T$  can be neglected in numerical calculation. We see that  $E_{st}$  is positive.



Now  $E_s$  and  $E_t$  can be calculated numerically except for the small terms included in  $w_s$  and  $w_t$  which depend on  $c_h$ . The result is not sensitive to any reasonable variation of  $c_h$  and consequently we use the value of  $c_h$  determined in (6.15.). The actual numerical calculation shows that both  $E_s$  and  $E_t$  are positive. In addition  $E_s > E_t$  on account of the term  $S$  in  $E_s$ . See equation (5.22.).

From (8.2.) we have

(8.8.)

$$\frac{S+T}{2} = N_\alpha - N_\beta$$

$$\frac{S-T}{2} = P_\alpha - P_\beta$$

Since  $N_\alpha$ ,  $N_\beta$ ,  $P_\alpha$  and  $P_\beta$  must be integers, the equations restrict the possible values of  $S$  and  $T$ . We have to distinguish between four different types of nuclei.

(1) The total number of neutrons is even and the total number of protons is even.

Here both  $N_\alpha + N_\beta$  and  $P_\alpha + P_\beta$  are even.

Thus both  $N_\alpha - N_\beta$  and  $P_\alpha - P_\beta$  are even.

Thus both  $(S+T)/2$  and  $(S-T)/2$  are even.

Thus both  $S$  and  $T$  are even.

The minimum of the terms in the energy (8.7.) due to the spin is obviously given by

$$S = T = 0.$$

(2) The total number of neutrons is odd and the total number of protons is odd.

Here both  $N_\alpha + N_\beta$  and  $P_\alpha + P_\beta$  are odd.

Thus both  $N_\alpha - N_\beta$  and  $P_\alpha - P_\beta$  are odd.

Thus both  $(S+T)/2$  and  $(S-T)/2$  are odd.

Thus both  $S$  and  $T$  are even.

Since  $E_s$  is greater than  $E_t$ , the minimum of the terms in the energy due to the spin (8.7.) is obviously given by

$$S = 0. \quad T = \pm 2.$$

(3) The total number of neutrons is even and the total number of ~~neutrons~~ protons is odd.

Here  $N_\alpha + N_\beta$  is even and  $P_\alpha + P_\beta$  is odd.

Thus  $N_\alpha - N_\beta$  is even and  $P_\alpha - P_\beta$  is odd.

Thus  $(S+T)/2$  is even and  $(S-T)/2$  is odd.

Thus  $S$  and  $T$  are both odd.



~~Since the term in  $E_{st}$  is positive, the minimum~~ of the terms in the energy due to the spin (8.7.) is obviously given by

$$S = -T = \pm 1.$$

(4) The total number of neutrons is odd and the total number of protons is even.

Here  $P_\alpha + P_\beta$  is even and  $N_\alpha + N_\beta$  is odd.

Thus  $P_\alpha - P_\beta$  is even and  $N_\alpha - N_\beta$  is odd.

Thus  $(S-T)/2$  is even and  $(S+T)/2$  is odd.

Thus  $S$  and  $T$  are both odd.

As in the previous type (3) the minimum is given by

$$S = T = \pm 1.$$

We see that the statistical model gives all nuclei with an even number of both neutrons and protons the spin zero in complete agreement with experimental results.

We also see that the statistical model gives all nuclei with an odd number of both neutrons and protons the spin zero. This is in complete disagreement with the experimental results. The only nuclei of this type are  ${}^1_1\text{H}^2$ ,  ${}^3_3\text{Li}^6$  and  ${}^7_7\text{N}^{14}$  which all have unit spin. (i.e.  $S = 2$ ).

As the statistical model cannot be expected to hold good for the small nuclei, this disagreement cannot be brought as evidence against the validity of the model. The experimental results indicate that if a power expansion was possible for small nuclei, then  $E_t$  would be greater than  $E_s$ . There is no reason for this inequality to hold for the larger nuclei and the statistical model developed here does give the opposite inequality.

We also see that the statistical model gives all nuclei with an odd number of total particles the spin  $\frac{1}{2}$ . Thus the statistical model fails to account for the fact that besides the spin  $\frac{1}{2}$ , odd nuclei can have greater spins. Experimentally the possible values of the spin of nuclei with an odd number of total particles can be  $\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{5}{2}$  or  $\frac{7}{2}$  or  $\frac{9}{2}$ .  ${}^{83}_{81}\text{Bi}^{209}$ ,  ${}^{49}_{47}\text{In}^{115}$  and  ${}^{41}_{41}\text{Nb}^{93}$  as far as the evidence is available, have the spin  $\frac{9}{2}$ . Experiments have not yet established the existence of nuclei with spin greater than  $\frac{9}{2}$ .

(d.1.) THE STABILITY BREADTH OF  $\Delta$  FOR NUCLEI WITH AN EVEN NUMBER OF NEUTRONS AND AN EVEN NUMBER OF PROTONS.

In discussing stability, we are comparing the energies of neighbouring nuclei whose isotopic number varies by two and of course both neighbouring nuclei have the same number of particles.

As we are here interested in a nucleus with an even number of neutrons and an even number of protons we start off with a nucleus of this type. For such a nucleus we have  $S = T = 0$ . The neighbouring nucleus with which we compare energies is now of the type (2) i.e. having an odd number of neutrons and an odd number of protons. For such a nucleus we have  $S = 0$ , and  $T = \pm 2$ .

Using the expression for the total energy obtained by substitution of equations (8.1.) in the expression (5.18.)\*, we obtain from (8.6.) and (8.8.) the stability conditions

$$E_0(\tau_0) + \frac{\Delta}{A} E_1(\tau_0) + \frac{\Delta^2}{A^2} \bar{E}_2(\tau_0) \\ < E_0(\tau_0) + \frac{\Delta-2}{A} E_1(\tau_0) + \frac{(\Delta-2)^2}{A^2} \bar{E}_2(\tau_0) + \frac{4E_t(\tau_0)}{A^2} + \mu$$

and

$$E_0(\tau_0) + \frac{\Delta}{A} E_1(\tau_0) + \frac{\Delta^2}{A^2} \bar{E}_2(\tau_0) \\ < E_0(\tau_0) + \frac{\Delta+2}{A} E_1(\tau_0) + \frac{(\Delta+2)^2}{A^2} \bar{E}_2(\tau_0) + \frac{4E_t(\tau_0)}{A^2} - \mu$$

Combining these two conditions we obtain the inequality

$$(8.9.) \quad -\frac{E_1 A}{2\bar{E}_2} + 1 + \frac{E_t}{\bar{E}_2} + \frac{A^2 \mu}{4\bar{E}_2} > \Delta > -\frac{E_1 A}{2\bar{E}_2} - 1 - \frac{E_t}{\bar{E}_2} + \frac{A^2 \mu}{4\bar{E}_2}$$

Thus the stability breadth of  $\Delta$  is

$$(8.10.) \quad 2 \left\{ 1 + \frac{E_t(\tau_0)}{\bar{E}_2(\tau_0)} \right\}.$$

\*

NOTE: Equation (5.18.) mentioned above refers to  $\tau$ . Actually we use equation (7.6.) which refers to  $\tau_0$ .

(d.2.) THE STABILITY BREADTH OF  $\Delta$  FOR NUCLEI WITH AN ODD NUMBER OF NEUTRONS AND AN ODD NUMBER OF PROTONS.

We have mentioned earlier in this chapter that only three such nuclei are known to be stable. Of course this result is well known and can be deduced from the saturation character of the nuclear forces. It is however interesting to see this ~~result~~ result derived from the statistical method.

As we are here interested in the nucleus with an odd total number of neutrons and an odd total number of protons, we start off with a nucleus of this type. For such a nucleus we have  $S' = 0$  and  $T = \pm 2$ . The neighbouring nucleus with which we compare energies is now of the type (1) i.e. having an even number of neutrons and an even number of protons. For such a nucleus we have  $S' = T = 0$ .

From (8.6.) and (8.8.) we now obtain the stability conditions

$$E_0 + \frac{\Delta}{A} E_1(r_0) + \frac{\Delta^2}{A^2} \bar{E}_2(r_0) + \frac{4E_t(r_0)}{A^2} < E_0(r_0) + \frac{\Delta-2}{A} E_1(r_0) + \frac{(\Delta-2)^2}{A^2} \bar{E}_2(r_0) + \mu$$

and

$$E_0(r_0) + \frac{\Delta}{A} E_1(r_0) + \frac{\Delta^2}{A^2} \bar{E}_2(r_0) + \frac{4E_t(r_0)}{A^2} < E_0(r_0) + \frac{\Delta+2}{A} E_1(r_0) + \frac{(\Delta+2)^2}{A^2} \bar{E}_2(r_0) - \mu$$

Combining these two conditions we obtain the inequality

$$(8.11.) \quad -\frac{E_1 A}{2\bar{E}_2} + 1 - \frac{E_t}{\bar{E}_2} + \frac{A^2 \mu}{4\bar{E}_2} > \Delta > -\frac{E_1 A}{2\bar{E}_2} - 1 + \frac{E_t}{\bar{E}_2} + \frac{A^2 \mu}{4\bar{E}_2}$$

Also we obtain in this case the stability breadth  $\Delta$  to be

$$(8.12.) \quad \Delta = 2 \left\{ 1 - \frac{E_t(r_0)}{\bar{E}_2(r_0)} \right\}$$

(d.3.) THE STABILITY BREADTH OF  $\Delta$  FOR NUCLEI  
WITH AN EVEN NUMBER OF NEUTRONS AND AN  
ODD NUMBER OF PROTONS.

As we are here interested in the nucleus with an even number of neutrons and an odd number of protons, we start off with a nucleus of this type. For such a nucleus we have  $S = -T = \pm 1$ . The neighbouring nucleus with which we compare energies is now of the type (4) i.e. having an odd number of neutrons and an even number of protons. For such a nucleus we have  $S = T = \pm 1$ . From (8.6.) and (8.8.) we obtain the stability conditions

$$E_0(r_0) + \frac{\Delta}{A} E_1(r_0) + \frac{\Delta^2}{A^2} \bar{E}_2(r_0) + \frac{1}{A^2} \{E_s(r_0) + E_t(r_0) - E_{st}(r_0)\} \\ < E_0(r_0) + \frac{\Delta-2}{A} E_1(r_0) + \frac{(\Delta-2)^2}{A^2} \bar{E}_2(r_0) + \frac{1}{A^2} \{E_s(r_0) + E_t(r_0) + E_{st}(r_0)\} + \mu$$

and

$$\bar{E}_0(r_0) + \frac{\Delta}{A} E_1(r_0) + \frac{\Delta^2}{A^2} \bar{E}_2(r_0) + \frac{1}{A^2} \{E_s(r_0) + E_t(r_0) - E_{st}(r_0)\} \\ < \bar{E}_0(r_0) + \frac{\Delta+2}{A} E_1(r_0) + \frac{(\Delta+2)^2}{A^2} \bar{E}_2(r_0) + \frac{1}{A^2} \{E_s(r_0) + E_t(r_0) + E_{st}(r_0)\} - \mu$$

Combining these two conditions we obtain the inequality

$$(8.13.) \quad -\frac{E_1 A}{2 \bar{E}_2} + 1 + \frac{A^2 \mu}{4 \bar{E}_2} + \frac{E_{st}}{2 \bar{E}_2} > \Delta > -\frac{E_1 A}{2 \bar{E}_2} - 1 + \frac{A^2 \mu}{4 \bar{E}_2} - \frac{E_{st}}{2 \bar{E}_2}$$

from which the spin terms except the negligible term in  $E_{st}$  have now dropped out.

The stability breadth is now

$$(8.14.) \quad 2 \left\{ 1 + \frac{E_{st}(r_0)}{2 \bar{E}_2(r_0)} \right\}$$

(d.4.) THE STABILITY BREADTH OF  $\Delta$  FOR NUCLEI WITH AN ODD NUMBER OF NEUTRONS AND AN EVEN NUMBER OF PROTONS.

As we are here interested in the nucleus with an odd number of neutrons and an even number of protons, we start off with a nucleus of this type. For such a nucleus we have  $S' = T = \pm 1$ . The neighbouring nucleus with which we compare energies is now of the type (3) i.e. having an even number of neutrons and an odd number of protons. For such a nucleus we have  $S = -T = \pm 1$ .

From (8.6.) and (8.8.) we now obtain the stability conditions

$$E_0(\tau_0) + \frac{\Delta}{A} E_1(\tau_0) + \frac{\Delta^2}{A^2} \bar{E}_2(\tau_0) + \frac{1}{A^2} \{ E_s(\tau_0) + E_t(\tau_0) + E_{st}(\tau_0) \} \\ < E_0(\tau_0) + \frac{\Delta-2}{A} E_1(\tau_0) + \frac{(\Delta-2)^2}{A^2} \bar{E}_2(\tau_0) + \frac{1}{A^2} \{ E_s(\tau_0) + E_t(\tau_0) - E_{st}(\tau_0) \} + \mu$$

and

$$E_0(\tau_0) + \frac{\Delta}{A} E_1(\tau_0) + \frac{\Delta^2}{A^2} \bar{E}_2(\tau_0) + \frac{1}{A^2} \{ E_s(\tau_0) + E_t(\tau_0) + E_{st}(\tau_0) \} \\ < E_0(\tau_0) + \frac{\Delta+2}{A} E_1(\tau_0) + \frac{(\Delta+2)^2}{A^2} \bar{E}_2(\tau_0) + \frac{1}{A^2} \{ E_s(\tau_0) + E_t(\tau_0) - E_{st}(\tau_0) \} - \mu$$

Combining these two conditions we obtain the inequality

$$(8.15.) \quad -\frac{E_1 A}{2 \bar{E}_2} + 1 + \frac{A^2 \mu}{4 \bar{E}_2} - \frac{E_{st}}{2 \bar{E}_2} > \Delta > -\frac{E_1 A}{2 \bar{E}_2} - 1 + \frac{A^2 \mu}{4 \bar{E}_2} + \frac{E_{st}}{2 \bar{E}_2}$$

The stability breadth is now

$$(8.16.) \quad 2 \left\{ 1 - \frac{E_{st}}{2 \bar{E}_2} \right\}$$

Since the stability breadths for  $\Delta$  are approximately equal for nuclei of types (3) and (4) we should expect that there are about equal numbers of both types of stable nuclei. This result is verified by the experimental data.



(e) RESULTS OBTAINED FROM THE DIFFERENT STABILITY BREADTHS .

For numerical calculation we introduce the values of  $\bar{E}_2$ ,  $E_t$  and  $E_{st}$  corresponding to a given atomic number. We find that the values of  $E_t$  and  $\bar{E}_2$  are approximately equal. Thus the ratio  $E_t/\bar{E}_2$  is approximately unity. Since the value of  $E_{st}$  is very small in comparison with  $\bar{E}_2$ , we can neglect the ratio  $E_{st}/\bar{E}_2$ .

From (8.10.) we see that the stability breadth is, for nuclei with an even number of both protons and neutrons, approximately four. Thus there is in general one or two stable nuclei of this type with given (even) atomic number. This is verified by the experimental data and there are only two exceptions ( $A = 102$  and  $124$ ) for which there exist three stable nuclei. These two exceptions cannot be explained by the statistical model. (It is possible that some other law of force than (4.15.) would give a stability breadth in the neighbourhood of eight, in which case it could account for the above mentioned two exceptions).

From (8.12.) we see that the stability breadth for nuclei with an odd number of both neutrons and protons is approximately zero. Thus there is no stable nuclei of this type with given (even) atomic number. As mentioned above, there are three exceptions to this rule but they have so small an atomic number that they do not come within the scope of the statistical treatment.

From (8.14.) we see that the stability breadth of nuclei with an even number of neutrons and an odd number of protons is very slightly greater than two. Since the difference of the isotopic numbers of neighbouring stable nuclei must be from the definitions greater than two, there exists only one stable nucleus of this type with given (odd) atomic number.

From (8.16.) we see that the stability breadth of nuclei with an odd number of neutrons and an even number of protons is very slightly less than two. Thus there exists only one stable nucleus of this type with given (odd) atomic number.

Actually there exists about twelve nuclei of an odd atomic number with two stable configurations, one of type (3) and the other of type (4). Since the difference of their isotopic numbers is only two, we cannot expect the statistical model to explain these exceptions.

Since the stability breadths for  $\Delta$  are approximately equal for nuclei of types (3) and (4) we should expect that there are about equal numbers of both types of stable nuclei. This result is verified by the experimental data.



## (f) THE ISOTOPIC NUMBERS.

Even atomic number:

Corresponding to any even value of the atomic number, we obtain from (8.9.) a maximum value of the isotopic number which we shall denote by  $\max \Delta$ , and a minimum value of the isotopic number which we shall denote by  $\min \Delta$ .

Graph V (page 85) gives both  $\max \Delta$  and  $\min \Delta$  plotted against the atomic number  $A$ . In addition all known stable nuclei of even atomic number have been indicated by a cross.

The values of the constants occurring in the statistical model were chosen in such a way that the stability condition for  $_{80}\text{Hg}^{200}$  was satisfied. We should therefore expect that the midpoint between the curves of  $\max \Delta$  and  $\min \Delta$  should have the value 40 for  $A = 200$ . Actually the mid-point is about  $1\frac{1}{2}$  units higher. Practically all of this discrepancy is due to the extra term involving the rest mass of the electron in (8.9.) which was not taken into account in calculating the constants. There we used the rougher stability condition  $dE/d\delta = 0$ . This accounts for 1.2, the extra  $\frac{1}{3}$  being probably due to the fact that we have used formulae in terms of  $\tau$  in calculating the constants whereas here we are using formulae in terms of  $\tau_0$ .

From the experimental results we would expect that the region of stability should be somewhat greater, as only a little over half of the stable nuclei of even atomic number do lie in the region between the two curves.

Odd atomic number:

Neglecting the small term  $E_{st}/\bar{E}_\lambda$ , both types (3) and (4) give the same expressions for  $\max \Delta$  and  $\min \Delta$ . Thus we can combine the two types.

Graph VI (page 86) gives both  $\max \Delta$  and  $\min \Delta$  plotted against the atomic number  $A$ . In addition all stable nuclei of odd atomic number have been indicated by a cross.

From the curves we see that the greatest number of stable nuclei of odd atomic number do lie in the region between the two curves.

As expected, the statistical model does not give the dips in the experimental curves (which can be visualised by looking at graphs V and VI).

The stable nuclei indicated by crosses are taken from the table in (G.1.).

## 9. APPLICATION OF THE STATISTICAL MODEL TO THE n-CURVES.

From the graphs of the previous chapter, we can in principle answer all questions about nuclear stability. For some purposes it is more convenient to consider the  $n$ -curves introduced by FUCHS (F.2.).

### (a) NOTATION.

For the purpose of this chapter it will be more convenient to introduce as in (F.2.) a different representation of the nucleus.

We shall represent the nucleus by the symbol

$$(m, n, p)$$

where  $m$  is the total number of alpha-particles which can be formed from the nucleus,  $n$  is the number of surplus neutrons and  $p$  is the number of surplus ~~neutro~~ protons. ( $p$  can only have the values 1 or 0 for otherwise another  $\alpha$ -particle can be formed from the nucleus.)

We have the following relations between  $m, n, p$  and  $N_\alpha, N_\beta, P_\alpha, P_\beta$ ;

$$(9.1.) \quad \begin{aligned} N_\alpha + N_\beta &= 2m + n \\ P_\alpha + P_\beta &= 2m + p \end{aligned}$$

We thus obtain

$$(9.2.) \quad \Delta = n - p$$

Thus the isotopic number has the value  $n$  or  $n-1$  according as  $p$  has the value 0 or 1.

The energy of the nucleus  $(m, n, p)$  will be represented by:-

$$E(m, n, p)$$

The energies used here are the ordinary energies and not the binding energies as used in (F.2.) and thus there are differences in the equations of this chapter from those of (F.2.).

(b) DEFINITIONS OF THE VARIOUS  $n_i$ .

Before defining the various  $n_i$  it is desirable to repeat the stability conditions.

The stability conditions against  $\beta^-$ -disintegration, (for the two cases when  $p$  is 0 or 1) are

$$(9.3.) \quad E(m, n, 0) < E(m, n-1, 1) + \mu$$

and

$$(9.4.) \quad E(m, n-1, 1) < E(m+1, n-4, 0) + \mu$$

The stability conditions against capture of a K-electron are really the same as (9.3.) and (9.4.). Thus, if (9.3.) holds, the nucleus  $(m, n-1, 1)$  is unstable and changes into the nucleus  $(m, n, 0)$  by capture of a K-electron (or by emission of a positron).

Corresponding to a given value of  $m$ , FUCHS then defines numbers  $n_i$ , where  $i = 1, 2, 3, 4$  as follows:-

$n_1$  is the solution of the equation

$$(9.5.) \quad E(m, n-1, 1) = E(m+1, n-4, 0) + \mu$$

when  $n$  is even.

$n_2$  is the solution of the equation

$$(9.6.) \quad E(m, n, 0) = E(m, n-1, 1) + \mu$$

when  $n$  is odd.

$n_3$  is the solution of equation (9.5.)  
when  $n$  is odd.

$n_4$  is the solution of equation (9.6.)  
when  $n$  is even.

(c) SIGNIFICANCE OF THE VARIOUS  $n_i$ .

From general assumptions about the nuclear forces, the following inequality is derived

(9.7.)

$$n_1 < n_2 < n_3 < n_4$$

The various  $n_i$  then divide the range of any arbitrary  $n$  into five portions. The composition of the stable nucleus depends in which portion it is. FUCHS has obtained the following typical scheme for the stable nuclei:-

$$\begin{array}{l}
 n < n_1 \quad \left\{ \begin{array}{ll} n = 2\gamma & (m, 2\gamma, 0) \\ n = 2\gamma + 1 & (m, 2\gamma + 1, 0) \end{array} \right. \quad \begin{array}{cc} \text{---} & \text{---} \\ \text{---} & \text{---} \end{array} \\
 \\
 n_1 < n < n_2 \quad \left\{ \begin{array}{ll} n = 2\gamma & (m, 2\gamma, 0) \\ n = 2\gamma + 1 & (m, 2\gamma + 1, 0) \end{array} \right. \quad \begin{array}{cc} \text{---} & (m+1, 2\gamma-4, 0) \\ \text{---} & \text{---} \end{array} \\
 \\
 n_2 < n < n_3 \quad \left\{ \begin{array}{ll} n = 2\gamma & (m, 2\gamma, 0) \\ n = 2\gamma + 1 & \text{---} \end{array} \right. \quad \begin{array}{cc} \text{---} & (m+1, 2\gamma-4, 0) \\ (m, 2\gamma, 1) & \text{---} \end{array} \\
 \\
 n_3 < n < n_4 \quad \left\{ \begin{array}{ll} n = 2\gamma & (m, 2\gamma, 0) \\ n = 2\gamma + 1 & \text{---} \end{array} \right. \quad \begin{array}{cc} \text{---} & (m+1, 2\gamma-4, 0) \\ \text{---} & (m+1, 2\gamma-3, 0) \end{array} \\
 \\
 n_4 < n \quad \left\{ \begin{array}{ll} n = 2\gamma & \text{---} \\ n = 2\gamma + 1 & \text{---} \end{array} \right. \quad \begin{array}{cc} \text{---} & (m+1, 2\gamma-4, 0) \\ \text{---} & (m+1, 2\gamma-3, 0) \end{array}
 \end{array}$$

This scheme is written in such a way that nuclei in the same row are isobars, and nuclei in the same column are isotopes.

If we do order the known stable nuclei in this way, we do obtain this type of scheme. Sometimes the regions are so small that they vanish entirely.

There also exist exceptions which are explained in (B.1.)

(d) VALUES OF  $n_1$  FROM THE STATISTICAL MODEL.

To find  $n_1$ , we use equation (9.5.). We note that  $(m, n-1, 1)$  is a nucleus of type (2) i.e. having an odd number of both protons and neutrons, whilst  $(m+1, n-4, 0)$  is a nucleus of the type (1) i.e. having an even number of both neutrons and protons. For the former nucleus we have  $S = 0$ ,  $T = \pm 2$ . For the latter nucleus we have  $S = T = 0$ .

Noting that  $A = 4m + n_1$ , we obtain

$$E_0 + \frac{E_1}{4m+n_1} (n_1-2) + \frac{\bar{E}_2 (n_1-2)^2}{(4m+n_1)^2} + \frac{4E_t}{(4m+n_1)^2}$$

$$= E_0 + \frac{E_1}{4m+n_1} (n_1-4) + \frac{\bar{E}_2 (n_1-4)^2}{(4m+n_1)^2} + \mu$$

i.e.

$$2E_1 (4m+n_1) + \bar{E}_2 (4n_1-12) + 4E_t = \mu (4m+n_1)^2$$

Splitting  $(4m+n_1)^2$  into the two factors  $A (4m+n_1)$ , we obtain a linear equation for  $n_1$ , of which the solution is

$$(9.8.) \quad n_1 = \frac{12\bar{E}_2 - 8mE_1 - 4E_t + 4mA\mu}{2E_1 + 4\bar{E}_2 - A\mu}$$

Here all terms are to the first order proportional to the atomic number  $A$ , so that the right hand side is practically independent of  $A$ . Thus this equation can easily be solved. For a given  $m$ , we assume a reasonable value of  $n_1$  and obtain  $A$  from  $A = 4m + n_1$ , and then  $n_1$  is calculated from (9.8.). By repetition of this process with the new value of  $n_1$ , we obtain better and better approximations to  $n_1$ . Since the right hand side changes very little with  $A$ , this repetition will not be necessary in practice.

To find  $n_2$ , we use equation (9.6.). We note that  $(m, n, 0)$  is a nucleus of type (4) i.e. having an odd number of neutrons and an even number of protons, whilst  $(m, n-1, 1)$  is a nucleus of type (3) i.e. having an even number of neutrons and an odd number of protons. For the former nucleus we have  $S = T = \pm 1$ . For the latter nucleus we have  $S = -T = \pm 1$ .

Noting that  $A = 4m + n_2$ , we obtain



$$E_0 + \frac{E_1 n_2}{4m+n_2} + \frac{\bar{E}_2 n_2^2}{(4m+n_2)^2} + \frac{E_s + E_t + E_{st}}{(4m+n_2)^2}$$

$$= E_0 + \frac{E_1 (n_2-2)}{4m+n_2} + \frac{\bar{E}_2 (n_2-2)^2}{(4m+n_2)^2} + \frac{E_s + E_t - E_{st}}{(4m+n_2)^2} + \nu$$

i.e.

$$2E_1(4m+n_2) + \bar{E}_2(4n_2-4) + 2E_{st} = \nu(4m+n_2)^2$$

Splitting  $(4m+n_2)^2$  into the two factors  $A(4m+n_2)$ , we obtain a linear equation for  $n_2$  of which the solution is

$$(9.9.) \quad n_2 = \frac{4\bar{E}_2 - 8mE_1 - 2E_{st} + 4m\nu}{2E_1 + 4\bar{E}_2 - \nu}$$

Similar remarks to those mentioned after equation (9.8.) also apply here.

In an exactly similar way we obtain expressions for  $n_3$  and  $n_4$  from equations (9.5.) and (9.6.) respectively. The expressions are

$$(9.10.) \quad n_3 = \frac{12\bar{E}_2 - 8mE_1 + 2E_{st} + 4m\nu}{2E_1 + 4\bar{E}_2 - \nu}$$

and

$$(9.11.) \quad n_4 = \frac{4\bar{E}_2 - 8mE_1 + 4E_t + 4m\nu}{2E_1 + 4\bar{E}_2 - \nu}$$



(e) The isotopic breadth of nuclei with an odd number of protons and an even number of neutrons.

From the scheme of stable nuclei on page 74, we see immediately the isotopic breadth of nuclei with an odd number of protons and an even number of neutrons is  $n_3 - n_2$ .

The values of  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  corresponding to a given value of  $m$  are given by (9.8.), (9.9.), (9.10.) and (9.11.) respectively. We have pointed out that the right hand side of these equations are expressions which change very little with  $A$ . We may therefore assume the same value of  $A$  in all four equations.

From equations (9.9.) and (9.10.) we immediately obtain

$$(9.12.) \quad n_3 - n_2 = \frac{8\bar{E}_2 + 4E_{st}}{2E_1 + 4\bar{E}_2 - A\mu}$$

The term in  $E_{st}$  is very small and can be neglected. Also since  $\mu$  is small we can neglect the term  $A\mu$ .

Numerical calculation shows that  $n_3 - n_2$  steadily increases from a value a little greater than 2 to a value  $2\frac{1}{2}$  as  $m$  increases.

Experimentally the number of stable isotopes of this type is always equal to one or two. Thus the experimental value of  $n_3 - n_2$  must lie between 2 and 4.

We can conclude that the agreement between the statistical model and experiment is here quite good.

(f) The isobaric region.

From the scheme of stable nuclei we see that the region in which stable ~~nuclei~~ <sup>isobars</sup> can exist is between  $n_1$  and  $n_4$ .

Bearing in mind the remarks of the last page we obtain

$$(9.13.) \quad n_4 - n_1 = \frac{8(E_t - \bar{E}_2)}{2E_1 + 4\bar{E}_2 - A\mu}$$

Numerical calculation shows that  $n_4 - n_1$  steadily decreases from about .2 to the negative value  $-.1$  as  $m$  increases passing through the value zero for about  $m = 30$ .

Approximately we can say that the statistical model gives the value zero to  $n_4 - n_1$ .

Experimentally the value of  $n_4 - n_1$  varies from 3 to 7.

We can conclude that the statistical model gives no agreement here with the experimental results.

This probably is due to the fact that the statistical model does not account satisfactorily for the energy due to the spin terms.

We can say that the statistical model does not sufficiently emphasise the difference between odd and even particles.

10. CONCLUSION.

Theoretical formulae have been developed for the energy of a nucleus according to a statistical model and an expansion has been made.

The results have been applied to the numerical calculation of energies, radii, the surplus of neutrons over protons in stable nuclei and finally to a discussion of the various  $n_i$ .

As seen from graph III, very good results have been obtained for the energy agreeing almost exactly with experimental results even down to the small nucleus  ${}^4\text{N}$  which really ought to be outside the scope of a statistical model, owing to having so few as 14 particles.

Graphs III and IV give good values for the mid-point between  $\max \Delta$  and  $\min \Delta$ . The breadths  $\max \Delta - \min \Delta$  ought to be somewhat larger in particular for the nuclei with an even number of particles.

The inclusion of the spin terms in the energy enables us to consider the stability of nuclei in greater detail than has been possible in the older calculations. We can state for any given nucleus, whether it ought to be stable or not. It is, of course, in the nature of the statistical model that local fluctuations are smoothed out, so that we can only expect statistical agreement with experimental results.

With this reservation the results are satisfactory for nuclei with an odd number of particles, except that the model only gives stable nuclei the spin one-half, whereas experiments show that nuclei exist with spins as great as nine-halves.

On the other hand, the region of stability for nuclei with an even number of particles as given by the model is too small, as can be seen from the fact that the total number of isobars could only just be two. (Or from  $n_4 - n_1 \sim 0$  which amounts to the same thing).

It is possible that with other interaction potentials better agreement could be obtained, it seems more probable that the disagreement is due to the fact that in the statistical model, the difference between even and odd particles is not sufficiently pronounced.

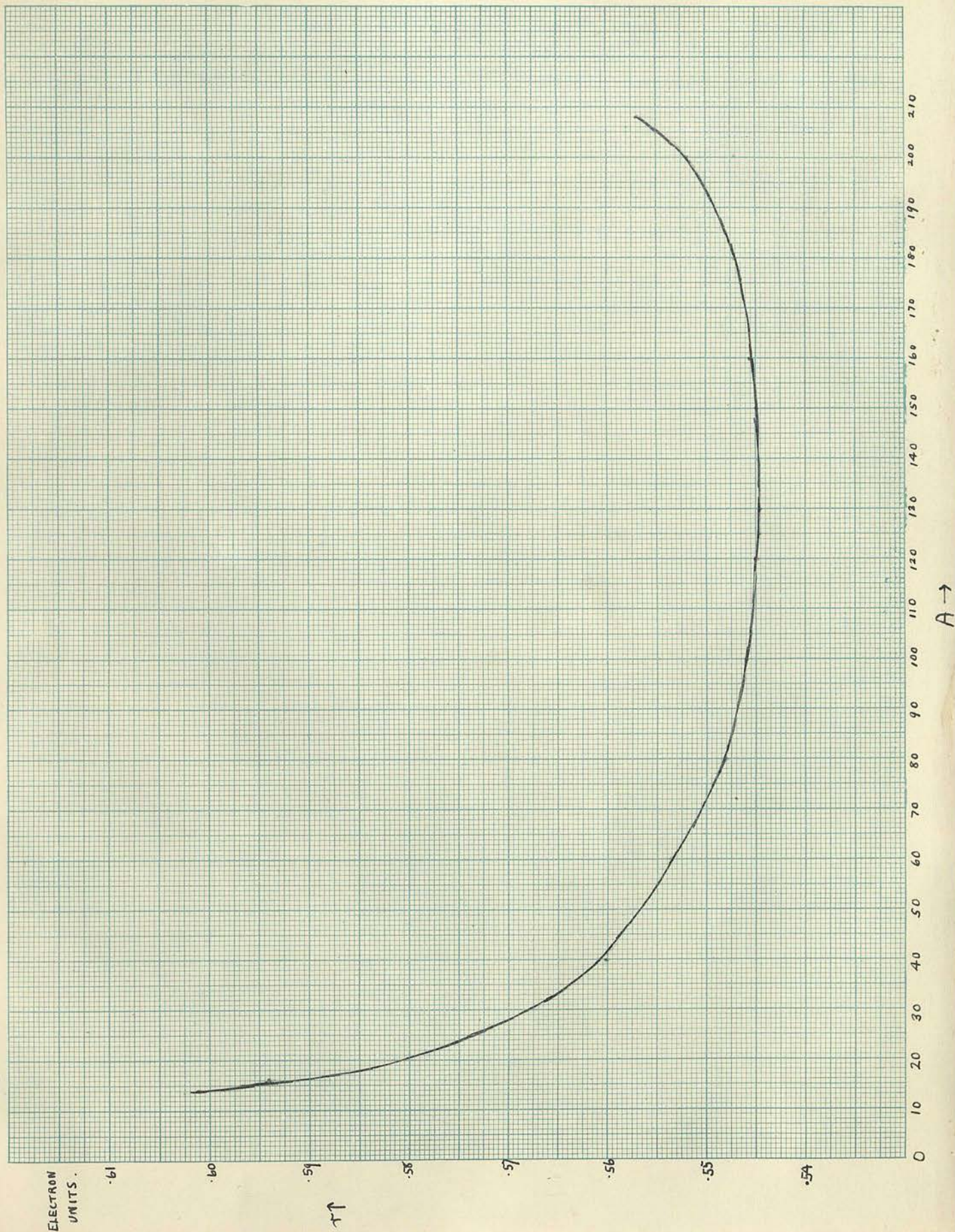
BIBLIOGRAPHY.

References to the following bibliography are indicated in the text by the initial and number on the left.

- (B.1.) BETHE and BACHER, Rev.Mod.Phys.8, 82 (1936).
- (B.2.) BETHE, Rev.Mod.Phys.9, 69 (1937).
- (F.1.) FROHLICH, HEITLER and KEMMER,  
Proc.Roy.Soc. 166, 154 (1938).
- (F.2.) FUCHS, Proc.Camb.Phil.Soc.35, 242 (1939).
- (G.1.) GREGOIRE, Jour.de Phys.et le Rad. Ser VII,  
9, 419 (1938).
- (H.1.) HARTREE, Proc.Camb.Phil.Soc.24, 89 (1928).
- (K.1.) KEMMER, Proc.Roy.Soc. 166, 127 (1938).
- (L.1.) LIVINGSTONE and BETHE, Rev.Mod.Phys.  
9, 245 (1937).
- (V.1.) VOLZ, Zeits.fur Phys.105, 537 (1937).
- (W.1.) WEIZSACKER, Die Atomkerne, (Leipsig 1937).
- (W.2.) WEIZSACKER, Zeits.fur Phys.96, 431 (1935).

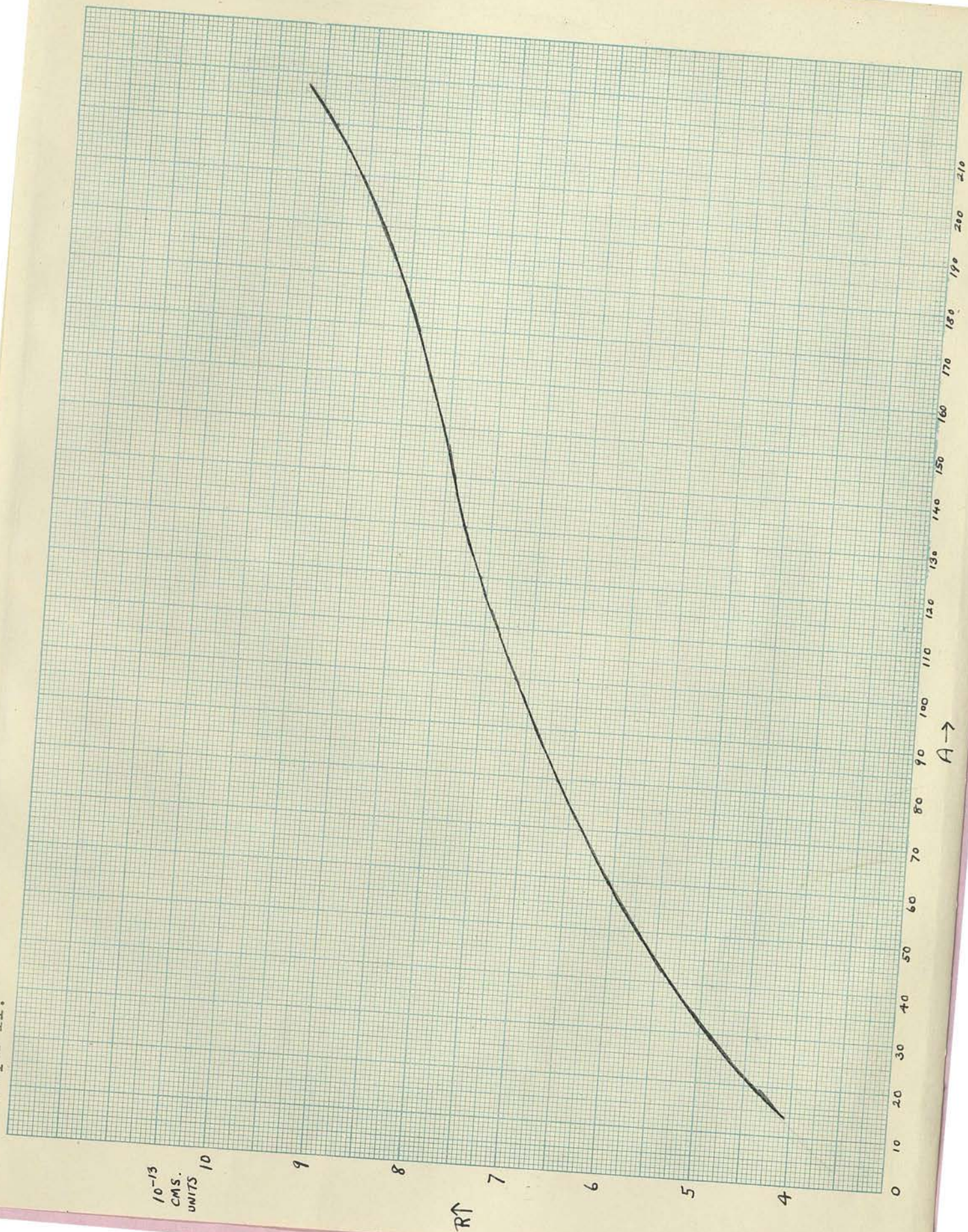


Graph I.



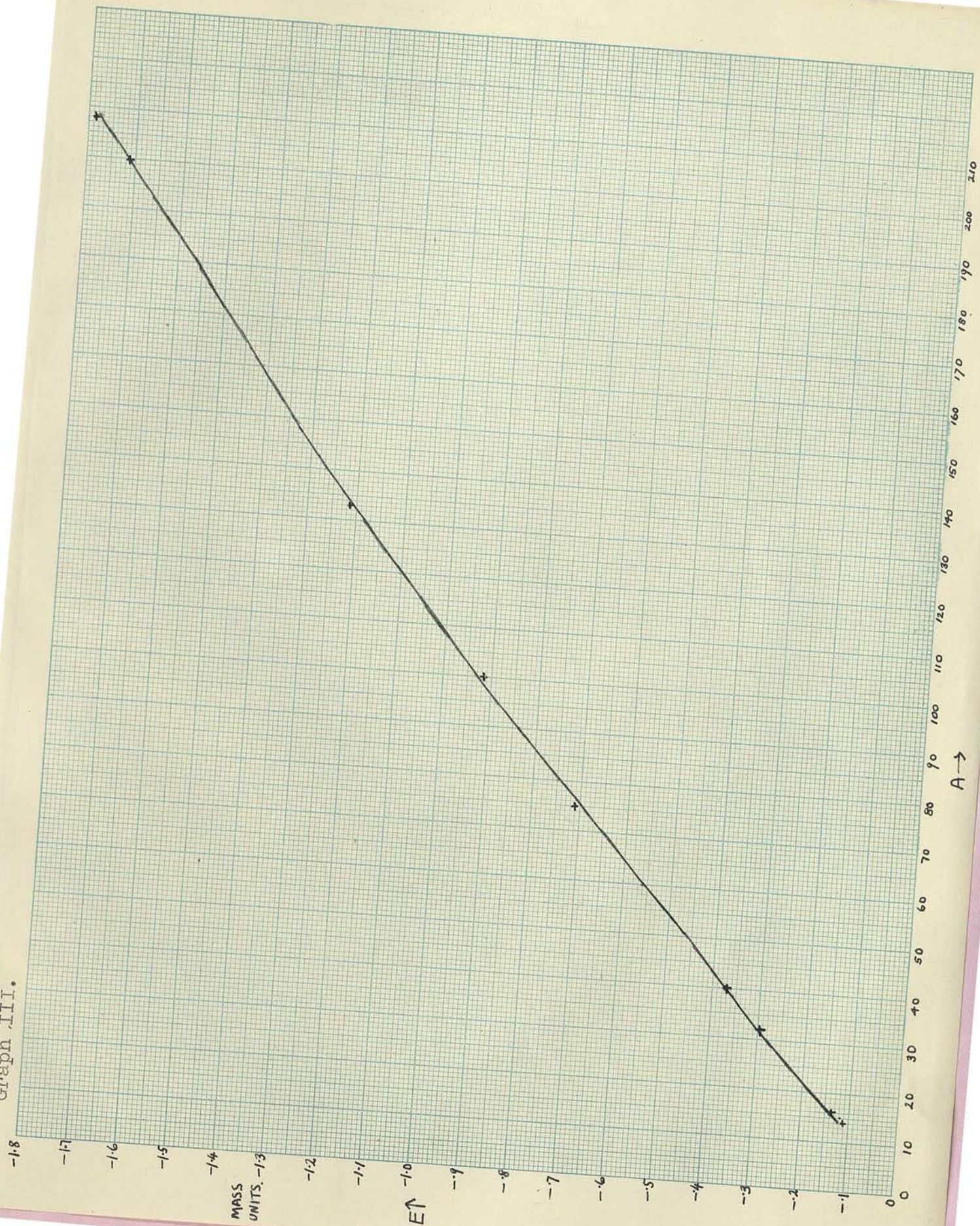


Graph II.



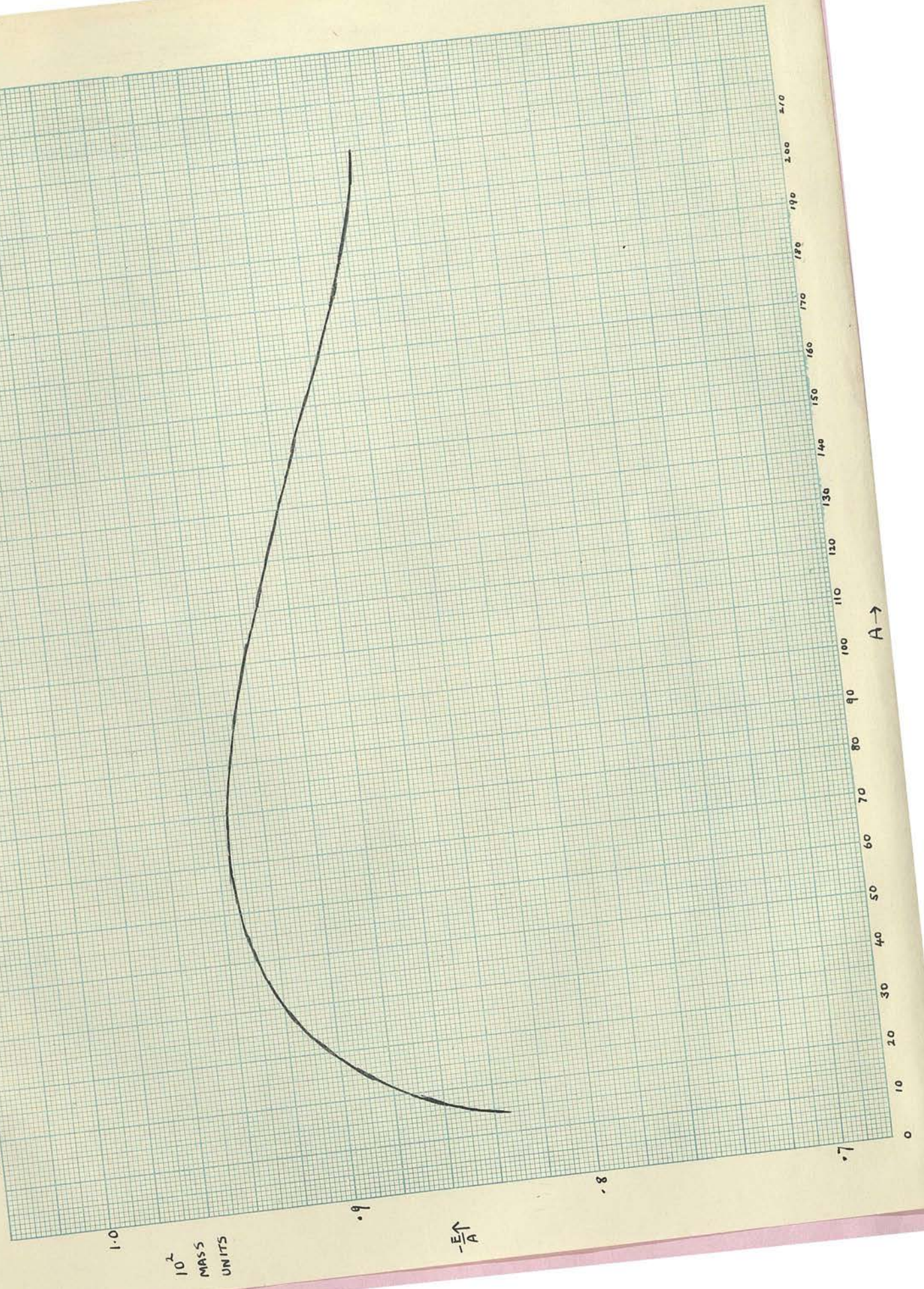


Graph III.



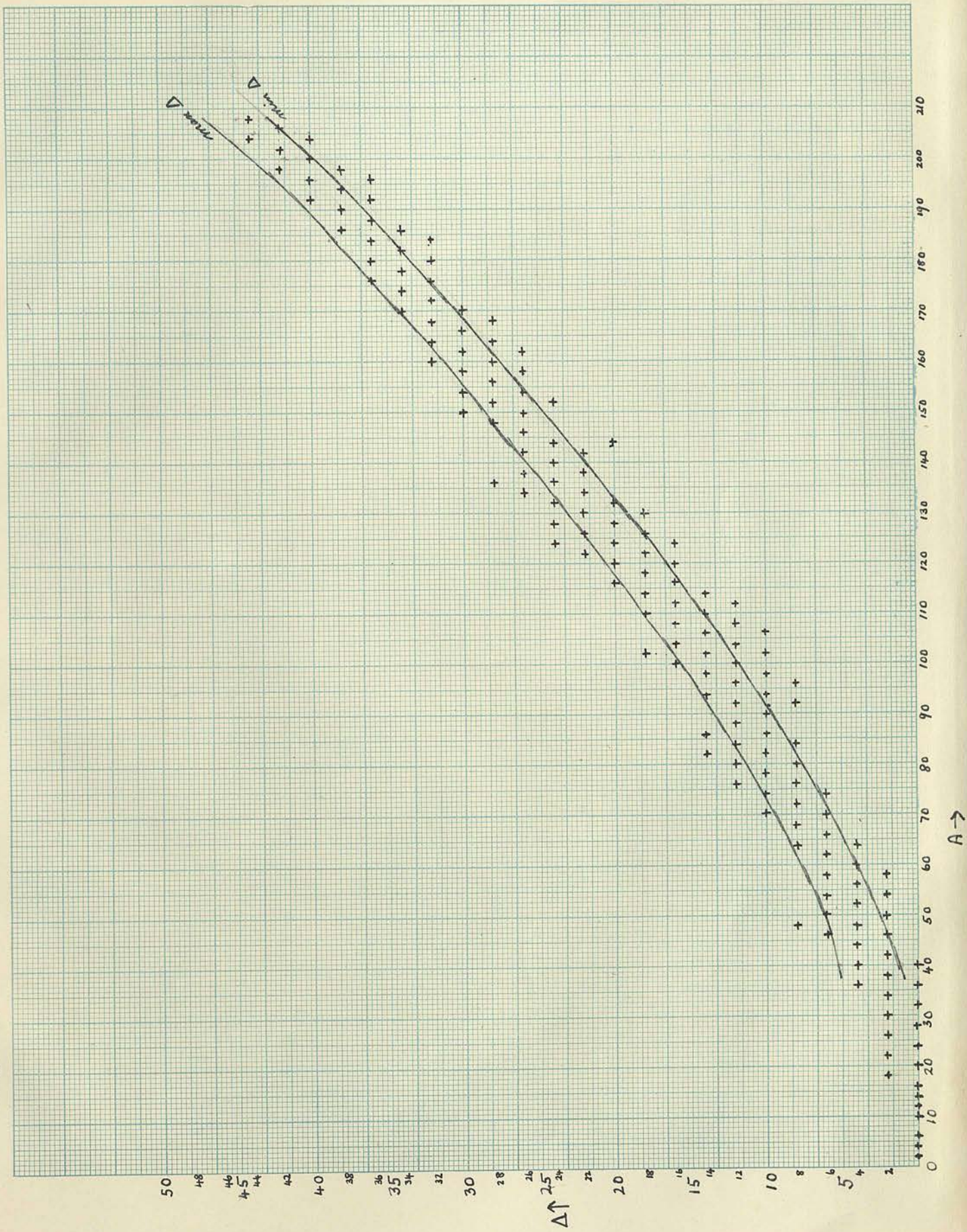


Graph IV.



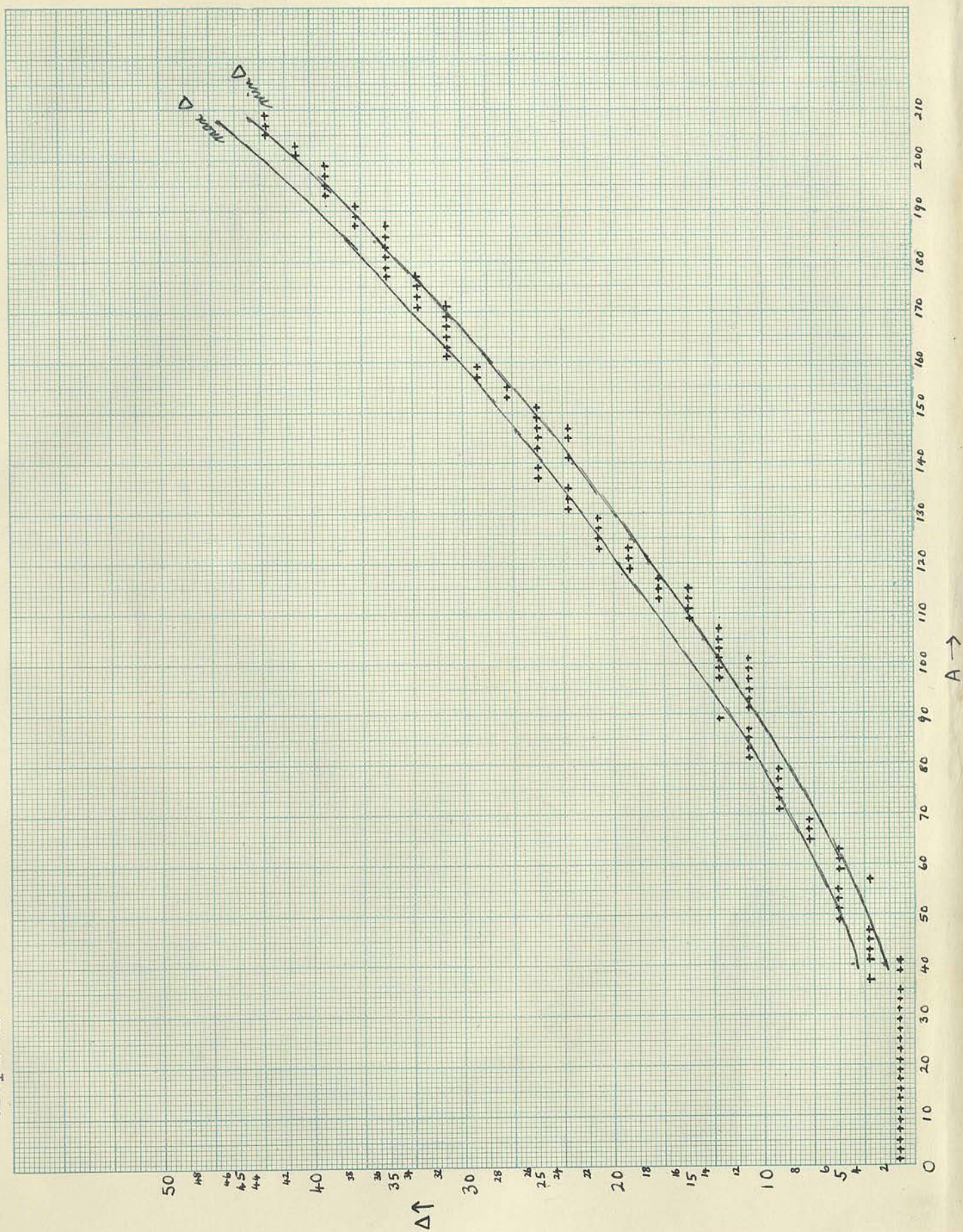


Graph V.





Graph VI





## APPENDIX A.

Here we proceed to calculate the integrals  $\varphi(x, y)$  and  $\psi(x, y)$  introduced in (4.19.). In order to do this introduce the integral

$$I_n = \int_0^\infty \frac{e^{-bt}}{t^n} (\sin pt - pt \cos pt) (\sin qt - qt \cos qt) dt$$

This integral reduces to  $\varphi(p, q)$  and  $\psi(p, q)$  when we put  $b=1$  and  $n=4$  and  $5$ . We have

$$\begin{aligned} I_0 &= \int_0^\infty e^{-bt} [\sin pt - pt \cos pt] [\sin qt - qt \cos qt] dt \\ &= \frac{1}{(2i)^2} \int_0^\infty e^{-bt} [e^{ipt}(1-ipt) - e^{-ipt}(1+ipt)] [e^{iqt}(1-iqt) - e^{-iqt}(1+iqt)] dt \\ &= \frac{1}{(2i)^2} \int_0^\infty e^{t(ip+iq-b)} (1-ipt)(1-iqt) dt - \frac{1}{(2i)^2} \int_0^\infty e^{t(ip-iq-b)} (1-ipt)(1+iqt) dt \\ &\quad - \frac{1}{(2i)^2} \int_0^\infty e^{t(-ip+iq-b)} (1+ipt)(1-iqt) dt + \frac{1}{(2i)^2} \int_0^\infty e^{t(-ip-iq-b)} (1+ipt)(1+iqt) dt \end{aligned}$$

Now

$$\begin{aligned} &\int_0^\infty e^{t[ip+iq-b]} (1-ipt)(1-iqt) dt \\ &= \left\{ \frac{e^{t(ip+iq-b)}}{ip+iq-b} [1-i(p+q)t - pq t^2] \right\}_0^\infty - \frac{1}{ip+iq-b} \int_0^\infty e^{t(ip+iq-b)} [-i(p+q) - 2pq t] dt \\ &= -\frac{1}{ip+iq-b} - \frac{1}{ip+iq-b} \left[ \left\{ \frac{e^{t(ip+iq-b)}}{ip+iq-b} [-i(p+q) - 2pq t] \right\}_0^\infty - \int_0^\infty \frac{e^{t(ip+iq-b)}}{ip+iq-b} (-2pq) dt \right] \\ &= -\frac{1}{ip+iq-b} - \frac{i(p+q)}{(ip+iq-b)^2} + \frac{1}{(ip+iq-b)^2} \left[ \frac{-2pq e^{t(ip+iq-b)}}{ip+iq-b} \right]_0^\infty \\ &= -\frac{1}{ip+iq-b} - \frac{i(p+q)}{(ip+iq-b)^2} + \frac{2pq}{(ip+iq-b)^3} \end{aligned}$$

Thus

$$\begin{aligned}
 (2i)^2 I_0 = & -\frac{1}{ip+iq-b} + \frac{1}{ip-iq-b} + \frac{1}{-ip+iq-b} + \frac{1}{ip+iq+b} \\
 & - \frac{i(p+q)}{(ip+iq-b)^2} + \frac{i(p-q)}{(ip-iq-b)^2} + \frac{i(-p+q)}{(-ip+iq-b)^2} + \frac{i(p+q)}{(ip+iq+b)^2} \\
 & + \frac{2pq}{(ip+iq-b)^3} + \frac{2pq}{(ip-iq-b)^3} + \frac{2pq}{(-ip+iq-b)^3} - \frac{2pq}{(ip+iq+b)^3}
 \end{aligned}$$

Integrating from infinity to  $b$ , with respect to  $b$ , we obtain

$$\begin{aligned}
 -(2i)^2 I_1 = & \log(b-ip-iq) + \log(b+ip+iq) \\
 & - \log(b-ip+iq) - \log(b+ip-iq) \\
 & + \frac{i(p+q)}{b-ip-iq} - \frac{i(p-q)}{b-ip+iq} + \frac{i(p-q)}{b+ip-iq} - \frac{i(p+q)}{b+ip+iq} \\
 & + \frac{pq}{(b-ip-iq)^2} + \frac{pq}{(b-ip+iq)^2} - \frac{pq}{(b+ip-iq)^2} + \frac{pq}{(b+ip+iq)^2}
 \end{aligned}$$

Integrating again we obtain

$$\begin{aligned}
 (2i)^2 I_2 = & b \log(b-ip-iq) + b \log(b+ip+iq) \\
 & - b \log(b-ip+iq) - b \log(b+ip-iq) \\
 & - \frac{pq}{b-ip-iq} - \frac{pq}{b+ip+iq} - \frac{pq}{b-ip+iq} - \frac{pq}{b+ip-iq}
 \end{aligned}$$

Integrating again we obtain

$$\begin{aligned}
 -2(2i)^2 I_3 = & (b^2 + p^2 + q^2) \left\{ \log(b-ip-iq) + \log(b+ip+iq) \right. \\
 & \left. - \log(b-ip+iq) - \log(b+ip-iq) \right\} \\
 & - 4pq
 \end{aligned}$$

Integrating again we obtain on reduction

$$\begin{aligned}
 -24I_4 = & [t^3 + 3(p^2 + q^2)t + 2i(p^3 + q^3)] \log(t + ip + iq) \\
 & + [t^3 + 3(p^2 + q^2)t - 2i(p^3 + q^3)] \log(t - ip - iq) \\
 & - [t^3 + 3(p^2 + q^2)t + 2i(p^3 - q^3)] \log(t + ip - iq) \\
 & - [t^3 + 3(p^2 + q^2)t - 2i(p^3 - q^3)] \log(t - ip + iq) \\
 & - 4pq t
 \end{aligned}$$

(A.1.)

Integrating for the last time we obtain on reduction

$$\begin{aligned}
 24I_5 = & \left[ \frac{t^4}{4} + \frac{3}{2}(p^2 + q^2)t^2 - \frac{3}{4}(p^2 - q^2)^2 + 2i(p^3 + q^3)t \right] \log(t + ip + iq) \\
 & + \left[ \frac{t^4}{4} + \frac{3}{2}(p^2 + q^2)t^2 - \frac{3}{4}(p^2 - q^2)^2 - 2i(p^3 + q^3)t \right] \log(t - ip - iq) \\
 & - \left[ \frac{t^4}{4} + \frac{3}{2}(p^2 + q^2)t^2 - \frac{3}{4}(p^2 - q^2)^2 + 2i(p^3 - q^3)t \right] \log(t + ip - iq) \\
 & - \left[ \frac{t^4}{4} + \frac{3}{2}(p^2 + q^2)t^2 - \frac{3}{4}(p^2 - q^2)^2 - 2i(p^3 - q^3)t \right] \log(t - ip + iq) \\
 & - pq t^2 + 3pq(p^2 + q^2).
 \end{aligned}$$

(A.2.)

Writing these expressions in their real form, and putting  $t=1$ , and  $m=4$  and 5 we obtain

$$\begin{aligned}
 24\psi(x, y) = & 4(x^3 + y^3) \tan^{-1}(x+y) - 4(x^3 - y^3) \tan^{-1}(x-y) \\
 & - [1 + 3(x^2 + y^2)] \log \frac{1 + (x+y)^2}{1 + (x-y)^2} + 4xy.
 \end{aligned}$$

$$\begin{aligned}
 24\varphi(x, y) = & \left\{ \frac{1}{4} + \frac{3}{2}(x^2 + y^2) - \frac{3}{4}(x^2 - y^2)^2 \right\} \log \frac{1 + (x+y)^2}{1 + (x-y)^2} \\
 & - 4(x^3 + y^3) \tan^{-1}(x+y) + 4(x^3 - y^3) \tan^{-1}(x-y) \\
 & - xy + 3xy(x^2 + y^2).
 \end{aligned}$$

# APPENDIX B.

Here we proceed to calculate the integrals  $\eta$  and  $\bar{\eta}$  introduced in (4.25.). We have

$$\int_0^R e^{-b\tau} d\tau = \left[ \frac{e^{-b\tau}}{-b} \right]_0^R = \frac{1 - e^{-bR}}{b}$$

Differentiating with respect to  $b$ , we obtain

$$\begin{aligned} - \int_0^R \tau e^{-b\tau} d\tau &= - \frac{1 - e^{-bR}}{b^2} + \frac{R e^{-bR}}{b} \\ &= - \frac{1}{b^2} [1 - bR e^{-bR} - e^{-bR}] \end{aligned}$$

Differentiating again with respect to  $b$ , we obtain

$$\begin{aligned} \int_0^R \tau^2 e^{-b\tau} d\tau &= \frac{2}{b^3} [1 - bR e^{-bR} - e^{-bR}] - \frac{1}{b^2} [-R e^{-bR} + bR^2 e^{-bR} + R e^{-bR}] \\ &= \frac{2}{b^3} \left[ 1 - e^{-bR} - bR e^{-bR} - \frac{1}{2} b^2 R^2 e^{-bR} \right] \end{aligned}$$

From these two integrals we immediately obtain

$$\eta = \frac{1}{a^2} \int_0^R \tau e^{-\tau/a} d\tau = 1 - e^{-R/a} (1 + R/a)$$

$$\bar{\eta} = \frac{1}{a^3} \int_0^R \tau^2 e^{-\tau/a} d\tau = 2 \left[ 1 - e^{-R/a} \left( 1 + R/a + \frac{1}{2} R^2/a^2 \right) \right]$$

From these equations we obtain

$$24 \varphi(x, x) = \left(\frac{1}{4} + 3x^2\right) \log(1+4x^2) - 8x^3 \tan^{-1} 2x - x^2 + 6x^4.$$

$$24 \varphi_x(x, y)_{x=y} = 3x \log(1+4x^2) - 12x^2 \tan^{-1} 2x + 12x^3$$

$$24 \varphi_{xx}(x, y)_{x=y} = 3(1-2x^2) \log(1+4x^2) - 24x \tan^{-1} 2x + 36x^2$$

$$24 \varphi_{xy}(x, y)_{x=y} = 6x^2 \log(1+4x^2)$$

and

$$24 \psi(x, x) = 8x^3 \tan^{-1} 2x - (1+6x^2) \log(1+4x^2) + 4x^2$$

$$24 \psi_x(x, y)_{x=y} = 12x^2 \tan^{-1} 2x - 6x \log(1+4x^2)$$

$$24 \psi_{xx}(x, y)_{x=y} = 24x \tan^{-1} 2x - 6 \log(1+4x^2) - 12x^2 \left[1 + \frac{1}{1+4x^2}\right]$$

$$24 \psi_{xy}(x, y)_{x=y} = 12x^2 \left[1 - \frac{1}{1+4x^2}\right]$$

Using the same notation for the functions  $\varphi(x, y)$  and  $\psi(x, y)$  as we have introduced for the function  $f(x, y)$  in (5.11.), (5.13.) and (5.23.) we obtain from the above results that:-

$$\begin{aligned} 24 \varphi_1 &= x^2 \left[ 3(1-2x^2) \log(1+4x^2) - 24x \tan^{-1} 2x + 36x^2 \right] \\ &\quad + x^2 \left[ 6x^2 \log(1+4x^2) \right] \\ &\quad - 2x \left[ 3x \log(1+4x^2) - 12x^2 \tan^{-1} 2x + 12x^3 \right] \end{aligned}$$

$$\begin{aligned} 24 \varphi_2 &= x^2 \left[ 3(1-2x^2) \log(1+4x^2) - 24x \tan^{-1} 2x + 36x^2 \right] \\ &\quad - x^2 \left[ 6x^2 \log(1+4x^2) \right] \\ &\quad - 2x \left[ 3x \log(1+4x^2) - 12x^2 \tan^{-1} 2x + 12x^3 \right] \end{aligned}$$



$$\begin{aligned}
 24 \psi_1 = & x^2 \left[ 24x \tan^{-1} 2x - 6 \log(1+4x^2) - 12x^2 \left(1 + \frac{1}{1+4x^2}\right) \right] \\
 & + x^2 \left[ 12x^2 \left(1 - \frac{1}{1+4x^2}\right) \right] \\
 & - 2x \left[ 12x^2 \tan^{-1} 2x - 6x \log(1+4x^2) \right]
 \end{aligned}$$

$$\begin{aligned}
 24 \psi_2 = & x^2 \left[ 24x \tan^{-1} 2x - 6 \log(1+4x^2) - 12x^2 \left(1 + \frac{1}{1+4x^2}\right) \right] \\
 & - x^2 \left[ 12x^2 \left(1 - \frac{1}{1+4x^2}\right) \right] \\
 & - 2x \left[ 12x^2 \tan^{-1} 2x - 6x \log(1+4x^2) \right] .
 \end{aligned}$$

i.e.

$$24 \varphi_1 = -3x^2 \log(1+4x^2) + 12x^4 .$$

$$24 \varphi_2 = -3x^2(1+4x^2) \log(1+4x^2) + 12x^4 .$$

$$24 \psi_1 = 6x^2 \log(1+4x^2) - \frac{24x^4}{1+4x^2} .$$

$$24 \psi_2 = 6x^2 \log(1+4x^2) - 24x^4 .$$

From equations similar to (5.23.) we obtain

$$24 \varphi_3 = 12x^4 \log(1+4x^2) .$$

$$24 \varphi_4 = -6x^2(1+2x^2) \log(1+4x^2) + 24x^4 .$$

$$24 \psi_3 = 24x^4 \left[ 1 - \frac{1}{1+4x^2} \right] .$$

$$24 \psi_4 = 12x^2 \log(1+4x^2) - 24x^4 \left[ 1 + \frac{1}{1+4x^2} \right] .$$

From these preceeding equations and from (4.20.) we obtain the following expressions for the  $f$ -functions:-

$$f_1 = -\frac{1}{8} x^2 \log(1+4x^2) + \frac{1}{2} x^4 \\ + k \left\{ \frac{1}{4} x^2 \log(1+4x^2) - \frac{x^4}{1+4x^2} \right\} .$$

$$f_2 = -\frac{1}{8} x^2 (1+4x^2) \log(1+4x^2) + \frac{1}{2} x^4 \\ + k \left\{ \frac{1}{4} x^2 \log(1+4x^2) - x^4 \right\} .$$

$$f_3 = \frac{1}{2} x^4 \log(1+4x^2) \\ + k \left\{ x^4 \left( 1 - \frac{1}{1+4x^2} \right) \right\} .$$

$$f_4 = -\frac{1}{4} x^2 (1+2x^2) \log(1+4x^2) + x^4 \\ + k \left\{ \frac{1}{2} x^2 \log(1+4x^2) - x^4 \left( 1 + \frac{1}{1+4x^2} \right) \right\} .$$

For completeness we may also note here that:-

$$f(x) = \frac{1}{96} (1+12x^2) \log(1+4x^2) - \frac{1}{3} x^3 \tan^{-1} 2x \\ - \frac{1}{24} x^2 + \frac{1}{4} x^4 \\ + k \left\{ \frac{1}{3} x^3 \tan^{-1} 2x - \frac{1}{24} (1+6x^2) \log(1+4x^2) \right. \\ \left. + \frac{1}{6} x^2 \right\} .$$

By straightforward differentiation we obtain the following expressions for the derivatives:-

$$f_1' = -\frac{1}{4}x \log(1+4x^2) - \frac{x^3}{1+4x^2} + 2x^3 \\ + k \left\{ \frac{1}{2}x \log(1+4x^2) - \frac{2x^3}{1+4x^2} + \frac{8x^5}{(1+4x^2)^2} \right\}.$$

$$f_2' = -\frac{1}{4}x(1+8x^2) \log(1+4x^2) - x + 2x^3 \\ + k \left\{ \frac{1}{2}x \log(1+4x^2) + \frac{2x^3}{1+4x^2} - 4x^3 \right\}.$$

$$f_3' = 2x^3 \log(1+4x^2) + \frac{4x^5}{1+4x^2} \\ + k \left\{ 4x^3 \left[ 1 - \frac{1}{1+4x^2} \right] + \frac{8x^5}{(1+4x^2)^2} \right\}.$$

$$f_4' = -\frac{1}{2}x(1+4x^2) \log(1+4x^2) - \frac{2x^3(1+2x^2)}{1+4x^2} + 4x^3 \\ + k \left\{ x \log(1+4x^2) - 4x^3 + \frac{8x^5}{(1+4x^2)^2} \right\}.$$

$$f'(x) = \frac{1}{4}x \log(1+4x^2) - x^2 \tan^{-1} 2x + x^3 \\ + k \left\{ x^2 \tan^{-1} 2x - \frac{1}{2}x \log(1+4x^2) \right\}.$$

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